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A thesis presented for the degree of
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Numerical Analysis Group
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Dedicated to
Thailand and my family.

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Abstract

A mathematical analysis has been carried out for a coupled pair of Cahn-Hilliard equations with a double well potential function with infinite walled free energy, which appears in modelling a phase separation on a thin film of binary liquid mixture coating substrate, which is wet by one component. Existence and uniqueness are proved for a weak formulation of the problem, which possesses a Lyapunov functional. Regularity results for the weak formulation are presented.

Semi and fully discrete finite element approximations are proposed where existence and uniqueness of their solutions are proven. Their convergence to the solution of the continuous solutions are presented. Error bound between semi-discrete and continuous solutions, between semi-discrete and fully discrete solutions, and between fully discrete and continuous solutions are all investigated. A practical algorithm to solve the fully discrete finite element formulation at each time step is introduced and its convergence is shown. Finally, a linear stability analysis of the equations in one dimension space is presented and some numerical simulations in one and two dimension spaces are preformed.

Declaration

The work in this thesis is based on research carried out at the Numerical Analysis Group, the Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction

Consider when a binary alloy, a composition u_m with components A and B, prepared at a uniform temperature T_I, greater than the critical temperature T_c, is deeply quenched to a temperature T_m less than T_c, the alloy being in stable or homogeneous state in one-phase region transforms into unstable state in two-phase region. In the transformation process, A and B components spontaneously separate and form domains pure in each component, u_a and u_b. This phenomenon is known as spinodal decomposition.

The phenomena of phase separation of a binary alloy first gained the interest of metallurgists. It was then noted that certain glass mixtures underwent the same type of transformation process. For further physical motivation for studying this problem we refer to Cahn & Hilliard [18], Cahn [17], and Cahn & Hilliard [19]. See also reviews by Gunton, San-Migul & Sahni [31] and Skripov & Skripov [47].

A phenomenological theory describing the above is provided by considering a free energy $\psi(u,T)$ where for $T > T_c$, $\psi_{uu}(u,T) > 0$ and for $T < T_c$, $\psi_{uu}(u,T) < 0$ in just one interval $[u_a^s,\ u_b^s]$ called the spinodal interval, see Figure 1.1, connected with the description is the phase diagram depicted in Figure 1.2. The spinodal curve β is the locus of points where $\psi_{uu}(u,T) = 0$. Above the coexistence curve, α, any uniform concentration is stable, see below. Below the spinodal curve the state (u_m, T_m) is unstable and the alloy separates into two values characterised by the values u_a and u_b where the line $T = T_m$ crosses the coexistence curve.

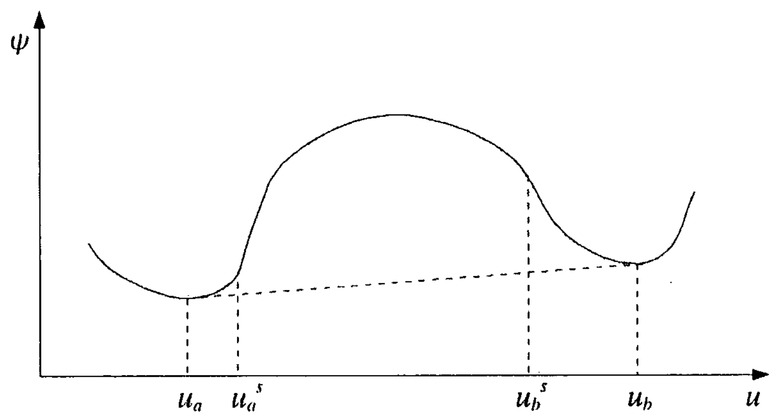


Figure 1.1: Free energy of the system below the critical temparature.

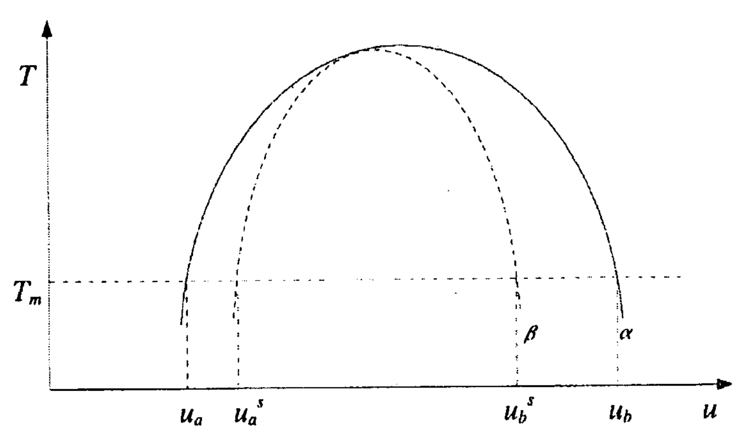


Figure 1.2: Phase diagram of binary alloy.

The Cahn-Hilliard equation, $\Omega \in \mathbb{R}^n$,

$$\frac{\partial u}{\partial t} - \Delta w = 0 \qquad x \in \Omega, t > 0, \tag{1.1.1a}$$

$$w = \psi'(u) - \gamma \Delta u \qquad x \in \Omega, t > 0, \tag{1.1.1b}$$

subject to the intitial condition

$$u(x,0) = u_0(x) \qquad x \in \Omega, \tag{1.1.1c}$$

and the boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{1.1.1d}$$

where u is the concentration, w is the chemical potential, $\psi(u)$ is the homogeneous free energy, and γ is a given positive constant relating to surface tension, has been proposed as a mathematical model representing spinodal decomposition of a binary alloy. For a mathematical discussion of the Cahn-Hilliard mathematical model of binary alloys see Elliott [25], Copetti and Elliott [22], Novick-Cohen [44], and Fife [29]. In order to model the surface energy of the interface separating the phases, Cahn and Hilliard [18] represented the free energy by adding a gradient term $\frac{\gamma}{2}|\nabla u|^2$ to a homogeneous free energy term $\psi(u)$ so that the free energy F of a binary alloy becomes

$$F(u) = \psi(u) + \frac{\gamma}{2}|\nabla u|^2. \tag{1.1.2}$$

The chemical potential w is the functional derivative of the total free energy $\mathcal{E}$

$$\mathcal{E}(u) = \int_\Omega F(u)dx = \int_\Omega \psi(u)\,dx + \frac{\gamma}{2}\int_\Omega |\nabla u|^2\,dx. \tag{1.1.3}$$

A prototype homogeneous free energy, used in Cahn and Hilliard [18], is

$$\psi(u) := \psi_L(0,T) + \frac{kT}{2}\left[(1+u)\log_e(1+u) + (1-u)\log_e(1-u)\right] - \frac{kT_c u^2}{2}, \tag{1.1.4}$$

see Figure 1.3, where u is the concentration varying between the values ± 1 which correspond to either atoms of type A or B, k is Boltzmann's constant, $0 < T < T_c$ is temparature considered as a parameter, the $\psi_L(0,T)$ is continuous in T, with

$$\psi_L(0,T) = \frac{kT_c}{2} - kT\log_e 2. \tag{1.1.5}$$

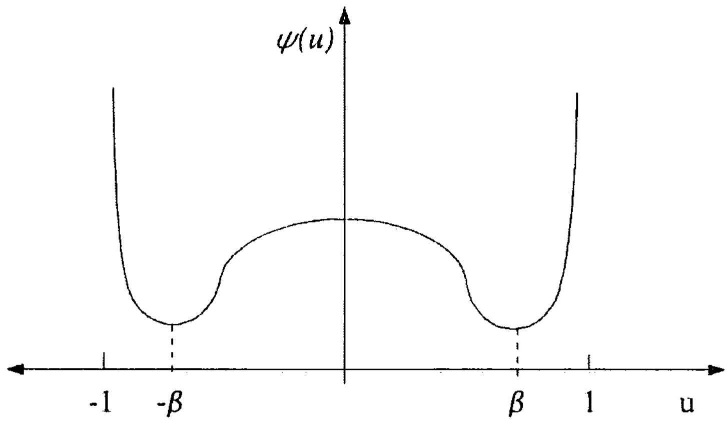


Figure 1.3: Non-differentiable homogeneous free energy in (1.1.4).

Since

$$\psi_L'(u) = -kT_c u + \frac{1}{2}kT\log_e\left[\frac{1+u}{1-u}\right], \text{ and } \psi_L''(u) = -kT_c + \frac{kT}{1-u^2}, \qquad (1.1.6)$$

it follows that, for $T > T_c$, ψ_L is a convex function on $(-1, 1)$ and, for $T < T_c$, ψ has the double well form. Furthermore, when $\psi_L''(.)$ is negative, the spinodal interval is $(-(1-\frac{T}{T_c})^{\frac{1}{2}}, (1-\frac{T}{T_c})^{\frac{1}{2}})$. Let the two minima of $\psi_L(u)$ be at $u_{T,+}$ and $u_{T,-} = -u_{T,+}$ as $\psi_L'(u)$ is an odd function. As we decrease T, the graph of $\frac{1}{2}kT\log_e\left(\frac{1+u}{1-u}\right)$ will be attracted to the u-axis. However, for T fixed, we note that

$$\lim_{u\to 1^-} \log_e\left(\frac{1+u}{1-u}\right) = +\infty, \qquad (1.1.7)$$

thus the graphs of $\frac{1}{2}kT\log_e\left(\frac{1+u}{1-u}\right)$ and $kT_c u$ will intersect at $u_{T,+}$ (and $u_{T,-}$) which will clearly converge to 1 (and -1) or

$$\lim_{T\to 0} u_{T,+} = 1 \quad \text{and} \quad \lim_{T\to 0} u_{T,-} = -1. \qquad (1.1.8)$$

Let Ω be bounded domain in $\mathbb{R}^d (d \leq 3)$ with Lipschitz boundary $\partial\Omega$. In this thesis, we consider the coupled pair of Cahn-Hilliard equation modelling a phase separation process on a thin film of binary liquid mixture coating substrate, which is wet by the two components of the alloy denoted by A and B, for further physical phenomenon see Keblinski et al. [35],

Problem **(P1)** For $D > 0$, find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial u_1}{\partial t} - \Delta w_1 = 0 \qquad \text{in } \Omega,\ t > 0, \qquad (1.1.9a)$$

$$\frac{\partial u_2}{\partial t} - \Delta w_2 = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.1.9b}$$

subject to the intial conditions

$$u_1(x,0) = u_1^0(x), \quad u_2(x,0) = u_2^0(x) \qquad \text{on } \Omega, \tag{1.1.9c}$$

and the boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \tag{1.1.9d}$$

where

$$w_1 = \frac{\delta F(u_1, u_2)}{\delta u_1}, \tag{1.1.9e}$$

$$w_2 = \frac{\delta F(u_1, u_2)}{\delta u_2}, \tag{1.1.9f}$$

$$F(u_1, u_2) = \psi_q(u_1) + c_1|\nabla u_1|^2 + \psi_q(u_2) + c_2|\nabla u_2|^2 + D\left(u_1 + \sqrt{\frac{a_1}{2b_1}}\right)^2 \left(u_2 + \sqrt{\frac{a_2}{2b_2}}\right)^2, \tag{1.1.9g}$$

$$\psi_q(u_1) = b_1 u_1^4 - a_1 u_1^2, \tag{1.1.9h}$$

$$\psi_q(u_2) = b_2 u_2^4 - a_2 u_2^2. \tag{1.1.9i}$$

The variable u_1 denotes a local concentration of A and B, u_2 indicates the presence of a liquid or a vapour phase. The function $F(u_1, u_2)$ is the free energy functional, and $\frac{\delta F(u_i,u_j)}{\delta u_i}$, for $i, j = 1, 2, i \neq j$, indicates the functional derivative. The constant c_i denotes the given surface tension of u_i. The coefficient a_i is proportional to $T_c - T$ of T_c and T in (1.1.3), and T_c represents the critical temperature of the liquid-vapour phase separation.

If $a_1 > 0, a_2 > 0$, there are two equilibrium phases for each field corresponding to $u_1 = \pm\sqrt{\frac{a_1}{2b_1}}$ and $u_2 = \pm\sqrt{\frac{a_2}{2b_2}}$, denoted by u_1^+, u_1^-, u_2^+ and u_2^-, respectively. The coupling D energetically inhibits the existence of the phase denoted by the (u_1^+, u_2^+). Thus we have a three-phase system: liquid A corresponds to (u_1^-, u_2^-) regions, liquid B to (u_1^+, u_2^-) regions and the vapour phase to (u_1^-, u_2^+) regions.

Taking $c_1 = c_2 = \frac{\gamma}{2}, \gamma > 0$, we obtain the gradient terms $\frac{\gamma}{2}|\nabla u_1|^2$ and $\frac{\gamma}{2}|\nabla u_2|^2$ in $F(u_1, u_2)$.

In the case where T is close to T_c, a Taylor series expansion shows that the approximation of (1.1.3) can be expressed as

$$\psi(u) \approx c(u^2 - u_c^2)^2 \tag{1.1.10}$$

which compares with $b_i u_i^4 - a_i u_i^2, i = 1, 2$ in (1.1.9h) and (1.1.9i).

As the minima of $b_i u_i^4 - a_i u_i^2$ are at $\pm\sqrt{\frac{a_i}{2b_i}}$, it is natural to replace

$$D\left(u_1 + \sqrt{\frac{a_1}{2b_1}}\right)^2 \left(u_2 + \sqrt{\frac{a_2}{2b_2}}\right)^2 \geq 0 \tag{1.1.11}$$

by

$$D(u_1 + 1)(u_2 + 1) \geq 0 \tag{1.1.12}$$

in the logarithmic case. This lead to the Cahn-Hilliard equation system with logarithmic homogeneous free energy as

Problem (**P2**) For $\gamma > 0$ and $D > 0$, find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial u_1}{\partial t} - \Delta w_1 = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.1.13a}$$

$$\frac{\partial u_2}{\partial t} - \Delta w_2 = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.1.13b}$$

subject to the intial conditions

$$u_1(x,0) = u_1^0(x), \quad u_2(x,0) = u_2^0(x) \qquad \text{on } \Omega, \tag{1.1.13c}$$

and the boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \tag{1.1.13d}$$

where

$$w_1 = \frac{\delta F(u_1, u_2)}{\delta u_1}, \tag{1.1.13e}$$

$$w_2 = \frac{\delta F(u_1, u_2)}{\delta u_2}, \tag{1.1.13f}$$

$$\begin{aligned} F(u_1, u_2) & \\ = \frac{T}{2}\Big[&(1+u_1)\ln(1+u_1) + (1+u_2)\ln(1+u_2) + (1-u_1)\ln(1-u_1) + (1-u_2)\ln(1-u_2)\Big] \\ &+ \frac{T_c}{2}\Big[(1-u_1^2) + (1-u_2^2)\Big] + \frac{\gamma}{2}\Big(|\nabla u_1|^2 + |\nabla u_2|^2\Big) + D(u_1+1)(u_2+1). \end{aligned} \tag{1.1.13g}$$

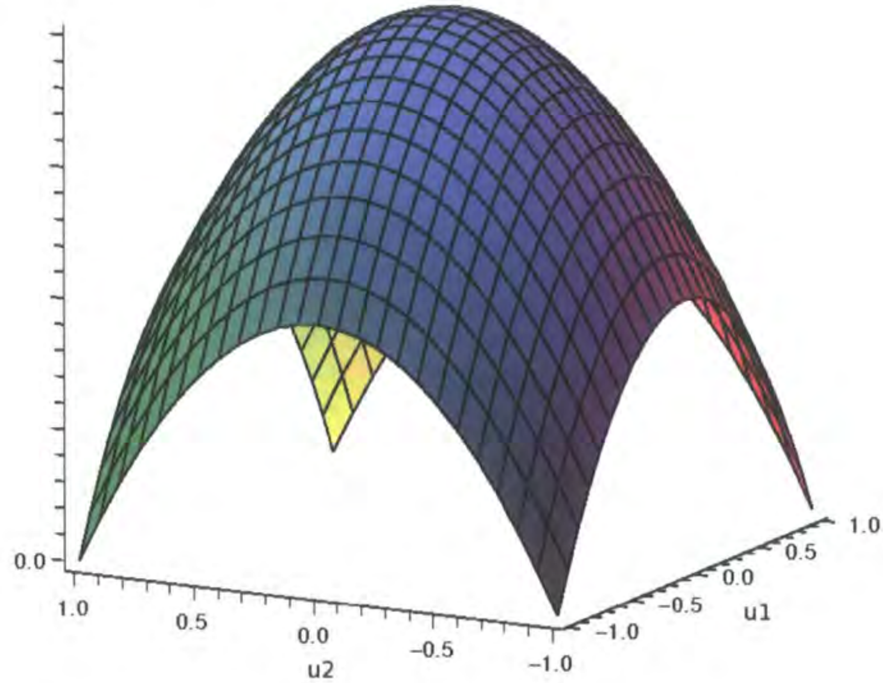


Figure 1.4: Homogeneous free energy under deep quench phase separation system.

Considering a deep quenched system, the temparature $T \to 0$, it is clear that $\frac{T}{T_c} \to 0$ the spinodal interval expands to $(-1,1)$, see Figure 1.4. The potential becomes an obstacle potential with homogeneous free energy

$$\psi(u) = \frac{T_c}{2}(1-u_1^2) + \frac{T_c}{2}(1-u_2^2) + I_{[-1,1]}(u_1) + I_{[-1,1]}(u_2), \tag{1.1.14}$$

where $I_{[-1,1]}(r)$ is the indicator function defined as

$$I_{[-1,1]}(r) = \begin{cases} 0 & \text{if } |r| \leq 1, \\ +\infty & \text{if } |r| > 1. \end{cases} \tag{1.1.15}$$

The homogeneous free energy in this form was proposed by Oono and Puri [45] who performed a numerical study of a discrete cell dynamical system and has been extensived used by Blowey and Elliott [13], Barrett and Blowey [3], and Nochetto, Verdi and Paolini [41]

Now taking $T_c = 1$, the Cahn-Hilliard equation system (**P2**) can be written as

Problem (**P3**) For $\gamma > 0$ and $D > 0$, find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial u_1}{\partial t} - \Delta w_1 = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.1.16a}$$

$$\frac{\partial u_2}{\partial t} - \Delta w_2 = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.1.16b}$$

where

$$w_1 \in \frac{\delta F(u_1, u_2)}{\delta u_1}, \tag{1.1.16c}$$

$$w_2 \in \frac{\delta F(u_1, u_2)}{\delta u_2}, \tag{1.1.16d}$$

$$\begin{aligned} F(u_1, u_2) &= \frac{1}{2}(1 - u_1^2) + \frac{1}{2}(1 - u_2^2) + \frac{\gamma}{2}|\nabla u_1|^2 + \frac{\gamma}{2}|\nabla u_2|^2 \\ &\quad + D(u_1 + 1)(u_2 + 1) + I_{[-1,1]}(u_1) + I_{[-1,1]}(u_2). \end{aligned} \tag{1.1.16e}$$

The Problem (**P3**) corresponds to the *deep quench* limit of Imran [33] where the free energy was for a model with a shallow quench. Together with this problem we include the following boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \tag{1.1.16f}$$

and initial conditions

$$u_1(x, 0) = u_1^0(x), \quad u_2(x, 0) = u_2^0(x) \qquad \text{on } \Omega, \tag{1.1.16g}$$

where ν is the unit normal pointing out of Ω.

Thus the Problem (**P3**) now is

<u>Problem (**P4**)</u> For $\gamma >$ and $D > 0$, find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial u_1}{\partial t} - \Delta w_1 = 0 \qquad \text{in } \Omega, t > 0, \tag{1.1.17a}$$

$$\frac{\partial u_2}{\partial t} - \Delta w_2 = 0 \qquad \text{in } \Omega, t > 0, \tag{1.1.17b}$$

$$w_1 \in \frac{\delta F(u_1, u_2)}{\delta u_1} = -\gamma \Delta u_1 - u_1 + D(u_2 + 1), \tag{1.1.17c}$$

$$w_2 \in \frac{\delta F(u_1, u_2)}{\delta u_2} = -\gamma \Delta u_2 - u_2 + D(u_1 + 1), \tag{1.1.17d}$$

subject to the intial conditions

$$u_1(x, 0) = u_1^0(x), \quad u_2(x, 0) = u_2^0(x) \qquad \text{on } \Omega, \tag{1.1.17e}$$

and the boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0 \qquad \text{on } \partial\Omega. \tag{1.1.17f}$$

If $D = 0$, then Problem **(P4)** reduces to two decoupled Cahn-Hilliard equations, which has been discussed in the mathematical literatures, see Elliott [25], Novick-Cohen [44], and Fife [29]. For this type of problem, we do not have liquid-vapour interfaces.

We now provide a layout of the contents of this thesis. In Chapter 2 a global existence theorem for a weak formulation possessing a Lyapunov functional is proven. Regularity results are presented for the weak formulation.

In Chapter 3 a semi-discrete finite element approximation is introduced. The existence and uniqueness is then proven. Stability bounds are obtained. Error bounds between the semi-discrete and continuous solutions are given as tools for the error analysis in Chapter 4.

A fully discrete finite element approximation is proposed in Chapter 4. It is shown that the scheme possesses a Lyapunov functional. An error bound between the discrete and continuous solutions is given by using the error bound results in Chapter 3.

In Chapter 5, a practical algorithm for solving the finite element problem at each time step is suggested and convergence of the algorithm is proven. Some interesting numerical simulations in one and two space dimensions are performed which show the expected behaviour of the physical problem.

Chapter 2

Evolutionary Problem

In Section 2.1 notation which is used in this chapter is defined. In Section 2.2 a weak formulation of the Cahn-Hiliard equation system is formulated and the existence and uniqueness is proven. In Section 2.3 regularity results for the weak solution are presented.

2.1 Notation and Results

Throughout Ω denotes an open bounded domain in $\mathbb{R}^d$, $(d = 1, 2, 3)$, we denote the norm of $H^p(\Omega)(p \geq 0)$ by $\|.\|_p$, the semi-norm $\|D^p\eta\|_0$ by $|\eta|_p$ and the $L^2(\Omega)$ inner-product by $(\cdot, \cdot)$. For $d = 2, 3$ we assume that $\partial\Omega$ is Lipschitz continuous.

We introduce the Green's operator $\mathcal{G}_N : \mathcal{F} \to V$ for the inverse of the Laplacian with zero Neumann boundary data :- given $f \in \mathcal{F} := \{f \in (H^1(\Omega))' : \langle f, 1\rangle = 0\}$ we define $\mathcal{G}_N f \in H^1(\Omega)$ to be the unique solution of

$$(\nabla \mathcal{G}_N f, \nabla \eta) = \langle f, \eta\rangle \qquad \forall \eta \in H^1(\Omega), \tag{2.1.1a}$$

$$(\mathcal{G}_N f, 1) = 0, \tag{2.1.1b}$$

where $V := \{v \in H^1(\Omega) : (v, 1) = 0\}$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. Note that

$$\langle f, \eta\rangle \equiv (f, \eta) \qquad \forall f \in L^2(\Omega). \tag{2.1.2}$$

The existence and uniqueness of $\mathcal{G}_N f$ follows from the Lax-Milgram theorem, the fact that the space $\mathcal{F}$ is linear, and the Poincaré inequality

$$|\eta|_0 \leq C_p\{|(\eta, 1)| + |\eta|_1\} \qquad \forall \eta \in H^1(\Omega). \tag{2.1.3}$$

For $f \in \mathcal{F}$ we define

$$\|f\|_{-1}^2 \equiv |\mathcal{G}_N f|_1^2 = (\nabla \mathcal{G}_N f, \nabla \mathcal{G}_N f) = \langle \mathcal{G}_N f, f \rangle, \tag{2.1.4}$$

and note that if $f \in \mathcal{F} \cap L^2(\Omega)$ then

$$\|f\|_{-1} = (\mathcal{G}_N f, f)^{\frac{1}{2}}. \tag{2.1.5}$$

For $f \in \mathcal{F} \cap L^2(\Omega)$, on noting (2.1.5), the Cauchy-Schwarz inequality, the Poincaré inequality (2.1.3) and (2.1.4) yields

$$\|f\|_{-1}^2 = (\mathcal{G}_N f, f) \leq |f|_0 |\mathcal{G}_N f|_0 \leq C_P |f|_0 |\mathcal{G}_N f|_1 = C_P |f|_0 \|f\|_{-1}, \tag{2.1.6}$$

so that

$$\|f\|_{-1} \leq C_P |f|_0. \tag{2.1.7}$$

Introducing the Young's inequality, for $\epsilon > 0$, a, $b \geq 0$, $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, formulated as

$$ab \leq \epsilon \frac{a^p}{p} + \epsilon^{\frac{-q}{p}} \frac{b^q}{q}. \tag{2.1.8}$$

We introduce also the Cauchy's inequality, a special case when $p = q = 2$ in (2.1.8), Rodrigues [46], expressed as

$$ab \leq \epsilon \frac{a^2}{2} + \epsilon^{-1} \frac{b^2}{2}. \tag{2.1.9}$$

We then use the Cauchy's inequality (2.1.9) to obtain, for all $\alpha > 0$,

$$\begin{aligned} (f, \eta)^h \equiv (\nabla \mathcal{G}_N f, \nabla \eta) &\leq |\mathcal{G}_N f|_1 |\eta|_1 \\ &= \|f\|_{-1} |\eta|_1 \\ &\leq \frac{1}{2\alpha} \|f\|_{-1} + \frac{\alpha}{2} |\eta|_1^2 \qquad \forall f \in \mathcal{F}^h, \eta \in H^1(\Omega). \end{aligned} \tag{2.1.10}$$

Let $\{z_j\}_{j=1}^{\infty}$ be the orthogonal basis for $H^1(\Omega)$ consisting of the eigenfunctions for

$$-\Delta z_j + z_j \;=\; \mu_j z_j \;; \qquad \frac{\partial z_j}{\partial \nu} \;=\; 0, \tag{2.1.11}$$

and normalised so that

$$(z_i, z_j) \;=\; \delta_{ij} \qquad \text{and} \qquad z_1 \;=\; \frac{1}{|\mu_1|^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}},\; \mu_1 \;=\; 1. \tag{2.1.12}$$

Note that z_j is the orthonormal basis for $L^2(\Omega)$. Let V^k denote the finite dimensional subspace of $H^1(\Omega)$ spanned by $\{z_j\}_{j=1}^k$.

We define the L^2 projection onto V^k, $P^k : L^2(\Omega) \to V^k$, by

$$P^k v \;=\; \sum_{j=1}^{k} (v, z_j) z_j, \tag{2.1.13}$$

We also notice that when $v \in H^1(\Omega)$, P^k is also the $H^1(\Omega)$ projection

$$(P^k v - v, \eta^k) \;=\; (\nabla(P^k v - v), \nabla \eta^k) \;=\; 0, \tag{2.1.14a}$$

$$\|P^k\|_{\mathcal{L}(H^1(\Omega),V^k)} \;=\; \|P^k\|_{\mathcal{L}(L^2(\Omega),V^k)} \;=\; 1. \tag{2.1.14b}$$

Consequently from the density of V^k in $H^1(\Omega)$ and compactness, it follows that $P^k v \to v$ in $L^2(\Omega)$.

Let $\eta^k \in V^k$ and set $\xi^k = P^k v + \eta^k \in V^k$. Using (2.1.14a) and the Cauchy-Schwarz inequality, we have, for $i = 0, 1$,

$$\begin{aligned}
|P^k v - v|_i^2 \;&=\; (P^k v - v, P^k v - v) + (P^k v - v, \eta^k) \\
&=\; (P^k v - v, P^k v - v + \eta^k) \\
&=\; (P^k v - v, \xi^k - v) \\
&\leq\; |P^k v - v|_i |\xi^k - v|_i.
\end{aligned}$$

Dividing by $|P^k v - v|_i$, we obtain

$$|P^k v - v|_i \;\leq\; |\xi^k - v|_i \qquad \forall \xi^k \in V^k, \quad i = 0, 1. \tag{2.1.15}$$

2.2 Existence and Uniqueness

Given, for $i = 1, 2$, $u_i^0 \in K := \{\eta \in H^1(\Omega) : -1 \leq \eta \leq 1\}$. We also define $K_{m_i} = \{\eta \in K : (\eta, 1) = (u_i^0, 1) = m_i\}$ with $m_i \equiv (u_i^0, 1) \in (-|\Omega|, |\Omega|)$. We consider two weak formulation of the Problem (**P4**) (1.1.17a-f) as Problem (**P**) and Problem (**Q**) defined as follows:

Problem (**P**): For $\gamma, D > 0$ find $\{u_1, u_2, w_1, w_2\}$ such that $u_i \in H^1(0,T;(H^1(\Omega))') \cap L^\infty(0,T;H^1(\Omega))$, $u_i \in K$, $w_i \in L^2(0,T;H^1(\Omega))$, for $i,j = 1,2$, $i \neq j$,

$$\left\langle \frac{\partial u_i}{\partial t}, \eta \right\rangle + (\nabla w_i, \nabla \eta) = 0 \qquad a.e.\ t \in (0,T),\ \forall \eta \in H^1(\Omega), \tag{2.2.1a}$$

$$\gamma(\nabla u_i, \nabla \eta - \nabla u_i) - (u_i, \eta - u_i) + D(u_j + 1, \eta - u_i) \geq (w_i, \eta - u_i)\ a.e.\ t \in (0,T) \quad \forall \eta \in K, \tag{2.2.1b}$$

$$u_i(\cdot, 0) = u_i^0(\cdot). \tag{2.2.1c}$$

Problem (**Q**): For $\gamma, D > 0$ find $\{u_1, u_2\}$ such that $u_i \in H^1(0,T;(H^1(\Omega))') \cap L^\infty(0,T;H^1(\Omega))$, $u_i \in K_{m_i}$ for $a.e.\ t \in (0,T)$ such that , for $i,j = 1,2$, $i \neq j$,

$$\gamma(\nabla u_i, \nabla \eta - \nabla u_i) + \left(\mathcal{G}_N \frac{\partial u_i}{\partial t}, \eta - u_i\right) - (u_i, \eta - u_i) + D(u_j + 1, \eta - u_i) \geq 0\ \forall \eta \in K_{m_i}, \tag{2.2.2a}$$

$$u_i(\cdot, 0) = u_i^0(\cdot) \in K_{m_i} \tag{2.2.2b}$$

with

$$w_i = -\mathcal{G}_N \frac{\partial u_i}{\partial t} + \frac{1}{|\Omega|}(w_i, 1). \tag{2.2.2c}$$

There is no abstract theory to which we can appeal to prove existence and uniqueness and in order that we can obtain the necessary regularity for the solution to prove the error bounds later we begin by regularizing Problem (**P**).

Let the homogeneous free energy with an obstacle potential be expressed as

$$\psi(r) = \frac{1}{2}(1 - r^2) + I_{[-1,1]}(r), \tag{2.2.3}$$

where $I_{[-1,1]}(r)$ is defined as (1.1.15). Given $0 < \epsilon < 1$ we introduce the regularizing

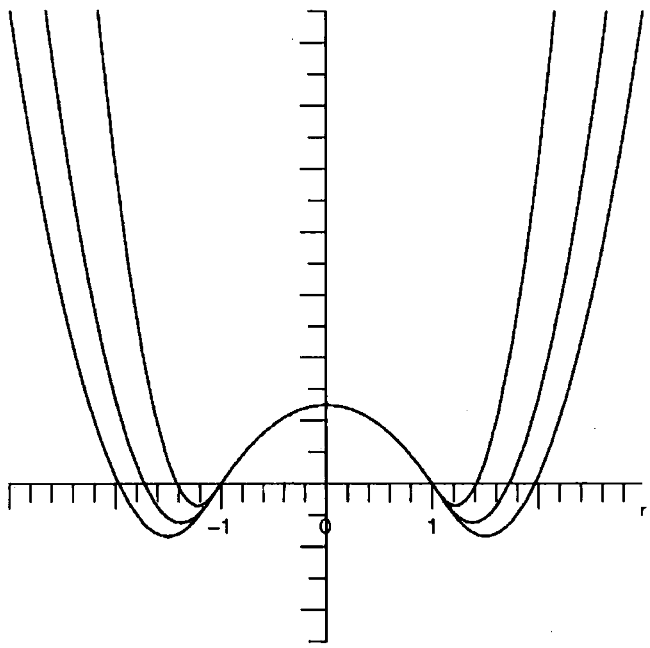


Figure 2.1: Penalised homogeneous free energy ψ_ϵ for three values of ϵ.

homogeneous free energy $\psi_\epsilon \in C^2(\mathbb{R})$ defined as follows: (see Figure 2.1)

$$\psi_\epsilon(r) := \begin{cases} \frac{1}{2\epsilon}\left(r-\left(1+\frac{\epsilon}{2}\right)\right)^2+\frac{1}{2}(1-r^2)+\frac{\epsilon}{24}, & \text{for } r \geq 1+\epsilon, \\ \frac{1}{6\epsilon^2}(r-1)^3+\frac{1}{2}(1-r^2), & \text{for } 1 < r < 1+\epsilon, \\ \frac{1}{2}(1-r^2), & \text{for } |r| \leq 1, \\ -\frac{1}{6\epsilon^2}(r+1)^3+\frac{1}{2}(1-r^2), & \text{for } -1-\epsilon < r < -1, \\ \frac{1}{2\epsilon}(r+(1+\frac{\epsilon}{2}))^2+\frac{1}{2}(1-r^2)+\frac{\epsilon}{24}, & \text{for } r \leq -1-\epsilon. \end{cases} \tag{2.2.4}$$

Differentiating (2.2.4) with respect to r we arrive at

$$\psi_\epsilon'(r) = \begin{cases} \frac{1}{\epsilon}\left(r-\left(1+\frac{\epsilon}{2}\right)\right)-r, & \text{for } r \geq 1+\epsilon, \\ \frac{1}{2\epsilon^2}(r-1)^2-r, & \text{for } 1 < r < 1+\epsilon, \\ -r, & \text{for } |r| \leq 1, \\ -\frac{1}{2\epsilon^2}(r+1)^2-r, & \text{for } -1-\epsilon < r < -1, \\ \frac{1}{\epsilon}\left(r+\left(1+\frac{\epsilon}{2}\right)\right)-r, & \text{for } r \leq -1-\epsilon. \end{cases} \tag{2.2.5}$$

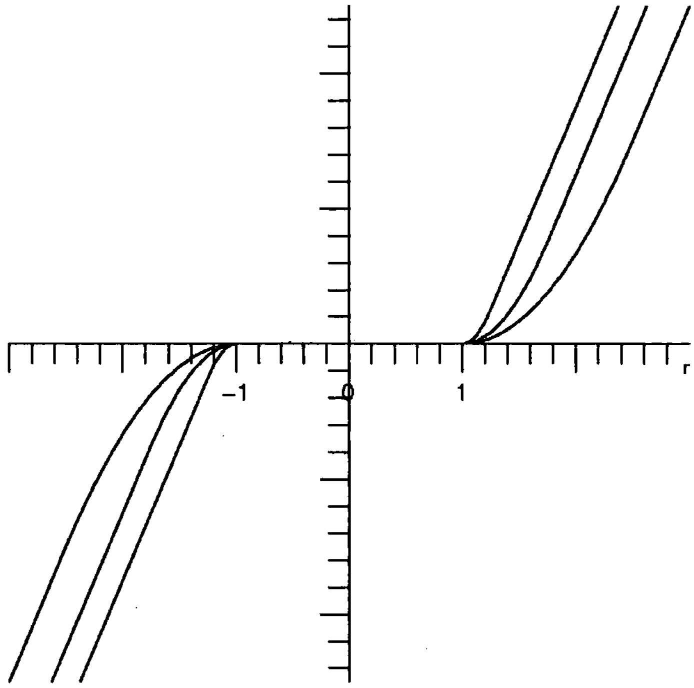


Figure 2.2: β_ϵ for three values of ϵ.

We define $\beta_\epsilon \in C^1(\mathbb{R})$ as follows: (see Figure 2.2)

$$\beta_\epsilon(r) \;:=\; \epsilon r + \epsilon \psi'_\epsilon(r) \;:=\; \begin{cases} r - \left(1 + \frac{\epsilon}{2}\right), & \text{for } r \geq 1 + \epsilon, \\ \frac{1}{2\epsilon}(r-1)^2, & \text{for } 1 < r < 1 + \epsilon, \\ 0, & \text{for } |r| \leq 1, \\ -\frac{1}{2\epsilon}(r+1)^2, & \text{for } -1-\epsilon < r < -1, \\ r + \left(1 + \frac{\epsilon}{2}\right), & \text{for } r \leq -1 - \epsilon. \end{cases} \tag{2.2.6}$$

We note that β_ϵ is a Lipschitz continuous function where

$$0 \;\leq\; \beta'_\epsilon \;\leq\; 1. \tag{2.2.7}$$

We also introduce the convex function

$$\psi_{\epsilon,1}(r) \;:=\; \psi_\epsilon(r) - \frac{1}{2}(1 - r^2), \tag{2.2.8}$$

so

$$\psi_{\epsilon,1}(r) = \frac{1}{2\epsilon}(\beta_\epsilon(r))^2 + \begin{cases} \frac{\epsilon}{24}, & \text{for } r \geq 1+\epsilon, \\ \frac{1}{6\epsilon^2}(r-1)^3 - \frac{1}{8\epsilon^3}(r-1)^4, & \text{for } 1 < r < 1+\epsilon, \\ 0, & \text{for } |r| \leq 1, \\ -\frac{1}{6\epsilon^2}(r+1)^3 - \frac{1}{8\epsilon^3}(r+1)^4, & \text{for } -1-\epsilon < r < -1, \\ \frac{\epsilon}{24}, & \text{for } r \leq -1-\epsilon, \end{cases} \tag{2.2.9}$$

which satisfies

$$\psi'_{\epsilon,1}(r) = \frac{1}{\epsilon}\beta_\epsilon(r) \quad \text{and} \quad 0 \leq \psi''_{\epsilon,1}(r) \leq \frac{1}{\epsilon}. \tag{2.2.10}$$

It follows from the definition of $\psi_{\epsilon,1}$ that

$$\psi_{\epsilon,1}(r) \geq \frac{1}{2\epsilon}|\beta_\epsilon(r)|^2, \quad \forall r, \tag{2.2.11}$$

and from (2.2.10) that

$$\psi_{\epsilon,1}(r) \geq \psi_{\epsilon,1}(s) + \frac{1}{\epsilon}\beta_\epsilon(s)(r-s). \tag{2.2.12}$$

Lemma 2.2.1 There exists a positive constant C_0, bounded independently of ϵ, such that, for $\epsilon < \frac{1}{8}\left(\frac{D}{2}+1\right)^{-1}$,

$$(\psi_\epsilon(u_1), 1) + (\psi_\epsilon(u_2), 1) + (D(u_1+1)(u_2+1), 1) \geq -C_0. \tag{2.2.13}$$

Proof: From the definition of β_ϵ, we have that

$$|\beta_\epsilon(r)| \leq |r+1| \quad \text{and} \quad |r+1| \leq |\beta_\epsilon(r)| + 2 + \frac{\epsilon}{2} \quad \forall r. \tag{2.2.14}$$

Noting (2.2.14) gives

$$\begin{aligned}
&\left|\int_\Omega (u_1+1)(u_2+1)\,dx\right| \\
&\quad \leq \int_\Omega |u_1+1||u_2+1|\,dx \\
&\quad \leq \int_\Omega \left(|\beta_\epsilon(u_1)| + 2 + \frac{\epsilon}{2}\right)\left(|\beta_\epsilon(u_2)| + 2 + \frac{\epsilon}{2}\right)\,dx \\
&\quad \leq \int_\Omega |\beta_\epsilon(u_1)||\beta_\epsilon(u_2)| + \left(2+\frac{\epsilon}{2}\right)|\beta(u_1)| + \left(2+\frac{\epsilon}{2}\right)|\beta(u_2)| + \left(2+\frac{\epsilon}{2}\right)^2\,dx \\
&\quad \leq \int_\Omega \left(\frac{1}{2}+\frac{1}{4\epsilon D}\right)|\beta_\epsilon(u_1)|^2 + \left(\frac{1}{2}+\frac{1}{4\epsilon D}\right)|\beta_\epsilon(u_2)|^2 + (2\epsilon D+1)\left(2+\frac{\epsilon}{2}\right)^2\,dx,
\end{aligned}$$

hence, as $D > 0$, it follows that

$$\begin{aligned}
&D\int_\Omega (u_1+1)(u_2+1)\,dx \\
&\geq -D\int_\Omega \left(\frac{1}{2}+\frac{1}{4\epsilon D}\right)|\beta_\epsilon(u_1)|^2+\left(\frac{1}{2}+\frac{1}{4\epsilon D}\right)|\beta_\epsilon(u_2)|^2+(2\epsilon D+1)(2+\frac{\epsilon}{2})^2\,dx.
\end{aligned} \tag{2.2.15}$$

Again noting (2.2.14) to obtain

$$\begin{aligned}
\left|\int_\Omega (1-u_1^2)\,dx\right| &= \left|\int_\Omega (1+u_1)\Big(2-(1+u_1)\Big)\,dx\right| \\
&\leq \int_\Omega |1+u_1||2+|1+u_1||\,dx \\
&\leq \int_\Omega \left(|\beta_\epsilon(u_1)|+2+\frac{\epsilon}{2}\right)\left(|\beta_\epsilon(u_1)|+4+\frac{\epsilon}{2}\right)\,dx \\
&= \int_\Omega |\beta_\epsilon(u_1)|^2+(6+\epsilon)|\beta_\epsilon(u_1)|+\left(2+\frac{\epsilon}{2}\right)\left(4+\frac{\epsilon}{2}\right)\,dx \\
&\leq \int_\Omega \left(1+\frac{1}{8\epsilon}\right)|\beta_\epsilon(u_1)|^2+2\epsilon(6+\epsilon)^2+\left(2+\frac{\epsilon}{2}\right)\left(4+\frac{\epsilon}{2}\right)\,dx.
\end{aligned} \tag{2.2.16}$$

In the same way as (2.2.16), we also have

$$\int_\Omega |1-u_2^2|\,dx \leq \int_\Omega \left(1+\frac{1}{8\epsilon}\right)|\beta_\epsilon(u_2)|^2+2\epsilon(6+\epsilon)^2+\left(2+\frac{\epsilon}{2}\right)\left(4+\frac{\epsilon}{2}\right)\,dx.$$

Then on noting (2.2.8), (2.2.9), (2.2.11) and (2.2.15) we obtain

$$\begin{aligned}
&\int_\Omega \Big[\psi_\epsilon(u_1)+\psi_\epsilon(u_2)\Big]\,dx+\int_\Omega D(u_1+1)(u_2+1)\,dx \\
&= \int_\Omega \left[\psi_{\epsilon,1}(u_1)+\frac{1}{2}(1-u_1^2)\right]\,dx+\int_\Omega \left[\psi_{\epsilon,1}(u_1)+\frac{1}{2}(1-u_1^2)\right]\,dx \\
&\quad +\int_\Omega D(u_1+1)(u_2+1)\,dx \\
&\geq \int_\Omega \left[\left(\frac{1}{8\epsilon}-\frac{D}{2}-1\right)|\beta_\epsilon(u_1)|^2\right]\,dx+\int_\Omega \left[\left(\frac{1}{8\epsilon}-\frac{D}{2}-1\right)|\beta_\epsilon(u_2)|^2\right]\,dx \\
&\quad -2\left[2\epsilon(6+\epsilon)^2+\left(2+\frac{\epsilon}{2}\right)\left(4+\frac{\epsilon}{2}\right)\right]|\Omega|+D(2\epsilon D+1)\left(2+\frac{\epsilon}{2}\right)^2|\Omega|.
\end{aligned}$$

Therefore, we have shown that (2.2.13) can be obtained for $\epsilon < \frac{1}{8}\left(\frac{D}{2}+1\right)^{-1}$, where $C_0 = 2\left[2\epsilon(6+\epsilon)^2+\left(2+\frac{\epsilon}{2}\right)\left(4+\frac{\epsilon}{2}\right)\right]|\Omega|+D(2\epsilon D+1)\left(2+\frac{\epsilon}{2}\right)^2|\Omega|.$ □

We now introduce the following penalised Problem $(\mathbf{P}_\epsilon)$ as

Problem ($\mathbf{P}_\epsilon$): For ϵ, $D > 0$ find $\{u_{\epsilon,1}, u_{\epsilon,2}, w_{\epsilon,1}, w_{\epsilon,2}\}$ such that $u_{\epsilon,i} \in H^1(0,T;(H^1(\Omega)')\cap L^\infty(0,T;H^1(\Omega))$ and $w_{\epsilon,i} \in L^2(0,T;H^1(\Omega))$, for $i,j = 1,2$, $i \neq j$,

$$\left\langle \frac{\partial u_{\epsilon,i}}{\partial t}, \eta \right\rangle + (\nabla w_{\epsilon,i}, \nabla \eta) = 0, \ a.e.\ t \in (0,T),\ \forall \eta \in H^1(\Omega), \tag{2.2.17a}$$

$$\gamma(\nabla u_{\epsilon,i}, \nabla\eta) + (\psi_\epsilon'(u_{\epsilon,i}),\eta) + D(u_{\epsilon,j}+1,\eta) = (w_{\epsilon,i},\eta),\ a.e.\ t \in (0,T)\ \forall \eta \in H^1(\Omega), \tag{2.2.17b}$$

$$u_{\epsilon,i}(\cdot,0) = u_i^0(\cdot). \tag{2.2.17c}$$

Theorem 2.2.2 There exists a unique solution to Problem ($\mathbf{P}_\epsilon$) such that:

$$\|u_{\epsilon,i}\|_{H^1(0,T;(H^1(\Omega))')} \leq C, \tag{2.2.18a}$$

$$\|u_{\epsilon,i}\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \tag{2.2.18b}$$

$$\|w_{\epsilon,i}\|_{L^2(0,T;H^1(\Omega))} \leq C\left(1 + T^{\frac{1}{2}}\right), \tag{2.2.18c}$$

where C is independent of ϵ and T. The Ginzburg-Landau free energy functional

$$\mathcal{E}_\gamma^\epsilon(u_{\epsilon,1},u_{\epsilon,2}) := \frac{\gamma}{2}|u_{\epsilon,1}|_1^2 + \frac{\gamma}{2}|u_{\epsilon,2}|_1^2 + (\psi_\epsilon(u_{\epsilon,1}),1) + (\psi_\epsilon(u_{\epsilon,2}),1) + D((u_{\epsilon,1}+1)(u_{\epsilon,2}+1),1) \tag{2.2.18d}$$

is a Lyapunov functional for Problem ($\mathbf{P}_\epsilon$).

Proof: We prove the existence using the Faedo-Galerkin method of Lions [37], see Appendix B. A Galerkin approximation to Problem ($\mathbf{P}_\epsilon$) is the following:

Find $\{u_{\epsilon,1}^k, u_{\epsilon,2}^k, w_{\epsilon,1}^k, w_{\epsilon,2}^k\} \in V^k \times V^k \times V^k \times V^k$, where

$$u_{\epsilon,i}^k(\mathbf{x},t) = \sum_{j=1}^k c_{i,j}(t) z_j(\mathbf{x}), \quad w_{\epsilon,i}^k(\mathbf{x},t) = \sum_{j=1}^k d_{i,j}(t) z_j(\mathbf{x})\ ; \quad i = 1,2,$$

such that

$$(\frac{\partial u_{\epsilon,i}^k}{\partial t}, \eta^k) + (\nabla w_{\epsilon,i}^k, \nabla \eta^k) = 0 \quad \forall \eta^k \in V^k, \tag{2.2.19a}$$

$$\gamma(\nabla u_{\epsilon,i}^k, \nabla\eta^k) + \frac{1}{\epsilon}(\beta_\epsilon(u_{\epsilon,i}^k),\eta^k) - (u_{\epsilon,i}^k,\eta^k) + D((u_{\epsilon,j}^k+1),\eta^k) = (w_{\epsilon,i}^k,\eta^k), \tag{2.2.19b}$$

$$u_{\epsilon,i}^k(0) = P^k u_i^0. \tag{2.2.19c}$$

Using (2.1.11), (2.1.12) and taking $\eta^k = z_j$ we can rewrite (2.2.19a,b) as, for $j = 1, 2, ..., k$,

$$\frac{dc_{1,j}(t)}{dt} + (\mu_j - 1)d_{1,j}(t) = 0, \tag{2.2.20a}$$

$$d_{1,j}(t) = \gamma(\mu_j - 1)c_{1,j}(t) + \frac{1}{\epsilon}(\beta_\epsilon(u^k_{\epsilon,1}), z_j) - c_{1,j}(t) + Dc_{2,j} + |\Omega|^{\frac{1}{2}}\delta_{i,j}, \tag{2.2.20b}$$

and

$$\frac{dc_{2,j}(t)}{dt} + (\mu_j - 1)d_{2,j}(t) = 0, \tag{2.2.21a}$$

$$d_{2,j}(t) = \gamma(\mu_j - 1)c_{2,j}(t) + \frac{1}{\epsilon}(\beta_\epsilon(u^k_{\epsilon,2}), z_j) - c_{2,j}(t) + Dc_{1,j} + |\Omega|^{\frac{1}{2}}\delta_{i,j}. \tag{2.2.21b}$$

This can be rewritten as a system of ODEs

$$\frac{d\underline{y}}{dt} = \underline{F}(\underline{y})$$

where $\underline{F}$ is uniformly Lipschitz. Hence we conclude the existence and uniqueness of $\underline{c}$, $\underline{d}$ on some time interval.

Since $\frac{\partial u^k_{\epsilon,i}}{\partial t} \in V^k$, differentiating $\mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}, u^k_{\epsilon,2})$ with respect to t we obtain

$$\begin{aligned}
&\frac{d\mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}, u^k_{\epsilon,2})}{dt} \\
&= \gamma\left(\nabla u^k_{\epsilon,1}, \nabla\frac{\partial u^k_{\epsilon,1}}{\partial t}\right) + \gamma\left(\nabla u^k_{\epsilon,2}, \nabla\frac{\partial u^k_{\epsilon,2}}{\partial t}\right) + \left(\psi'_\epsilon(u^k_{\epsilon,1}), \frac{\partial u^k_{\epsilon,1}}{\partial t}\right) + \left(\psi'_\epsilon(u^k_{\epsilon,2}), \frac{\partial u^k_{\epsilon,2}}{\partial t}\right) \\
&\quad + D\left(u^k_{\epsilon,2} + 1, \frac{\partial u^k_{\epsilon,1}}{\partial t}\right) + D\left(u^k_{\epsilon,1} + 1, \frac{\partial u^k_{\epsilon,2}}{\partial t}\right).
\end{aligned} \tag{2.2.22}$$

Using (2.2.19a,b), (2.2.4), (2.2.8), (2.2.10), (2.2.22) can be expressed as

$$\frac{d\mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}, u^k_{\epsilon,2})}{dt} = \left(w^k_{\epsilon,1}, \frac{\partial u^k_{\epsilon,1}}{\partial t}\right) + \left(w^k_{\epsilon,2}, \frac{\partial u^k_{\epsilon,2}}{\partial t}\right) = -|w^k_{\epsilon,1}|^2_1 - |w^k_{\epsilon,2}|^2_1. \tag{2.2.23}$$

In particular

$$\frac{d\mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}, u^k_{\epsilon,2})}{dt} \leq 0,$$

hence we have that

$$\mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}(s), u^k_{\epsilon,2}(s)) \geq \mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}(t), u^k_{\epsilon,2}(t)) \quad \forall s < t.$$

Therefore $\mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}, u^k_{\epsilon,2})$ is a Lyapunov functional.

Integrating (2.2.23) over $(0,t)$ we obtain

$$\mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}, u^k_{\epsilon,2}) = \mathcal{E}^\epsilon_\gamma(P^k u^0_1, P^k u^0_2) - \int_0^t |w^k_{\epsilon,1}(s)|^2_1 ds - \int_0^t |w^k_{\epsilon,2}(s)|^2_1 ds, \qquad (2.2.24)$$

thus

$$\mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}, u^k_{\epsilon,2}) + \int_0^t |w^k_{\epsilon,1}(s)|^2_1\, ds + \int_0^t |w^k_{\epsilon,2}(s)|^2_1\, ds = \mathcal{E}^\epsilon_\gamma(P^k u^0_1, P^k u^0_2). \qquad (2.2.25)$$

We now show that $\mathcal{E}^\epsilon_\gamma(P^k u^0_1, P^k u^0_2) \le C$.

On setting $s = P^k u^0_i$ and $r = u^0_i$ in (2.2.12), and noting (2.2.8), $|u^0_i| \le 1$, for $i = 1, 2$, it follows that

$$\begin{aligned}
\mathcal{E}^\epsilon_\gamma(P^k u^0_1, P^k u^0_2) &= \frac{\gamma}{2}|P^k u^0_1|^2_1 + \frac{\gamma}{2}|P^k u^0_2|^2_1 + (\psi_\epsilon(P^k u^0_1), 1), +(\psi_\epsilon(P^k u^0_2), 1) \\
&\quad + D((P^k u^0_1 + 1)(P^k u^0_2 + 1), 1) \\
&= \frac{\gamma}{2}|P^k u^0_1|^2_1 + \frac{\gamma}{2}|P^k u^0_2|^2_1 + (\psi_{\epsilon,1}(P^k u^0_1) + \frac{1}{2}(1 - (P^k u^0_1)^2, 1) \\
&\quad + (\psi_{\epsilon,1}(P^k u^0_2) + \frac{1}{2}(1 - (P^k u^0_2)^2, 1) + D((P^k u^0_1 + 1)(P^k u^0_2 + 1), 1) \\
&\le \frac{\gamma}{2}|P^k u^0_1|^2_1 + \frac{\gamma}{2}|P^k u^0_2|^2_1 + (\psi_{\epsilon,1}(u^0_1) - \frac{1}{\epsilon}(\beta_\epsilon(P^k u^0_1), (u^0_1 - P^k u^0_1)) \\
&\quad + \frac{1}{2}(1 - (P^k u^0_1)^2, 1) + (\psi_{\epsilon,1}(u^0_2) - \frac{1}{\epsilon}(\beta_\epsilon(P^k u^0_2), (u^0_2 - P^k u^0_2)) \\
&\quad + \frac{1}{2}(1 - (P^k u^0_2)^2, 1) + D((P^k u^0_1 + 1)(P^k u^0_2 + 1), 1). \qquad (2.2.26)
\end{aligned}$$

Since the Lipschitz continuity of β_ϵ and the strong convergence of $P^k u^0_i$ to u^0_i in $L^2(\Omega)$, for $i = 1, 2$,

$$\begin{aligned}
(\beta_\epsilon(P^k u^0_i), P^k u^0_i - u^0_i) &= (\beta_\epsilon(P^k u^0_i) - \beta_\epsilon(u^0_i), P^k u^0_i - u^0_i) + (\beta_\epsilon(u^0_i), P^k u^0_i - u^0_i) \\
&\le C(\epsilon)\|P^k u^0_i - u^0_i\|^2_0 \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

From (2.2.26) it follows that

$$\limsup_{k\to\infty} \mathcal{E}^\epsilon_\gamma(P^k u^0_1, P^k u^0_2) \le \mathcal{E}^\epsilon_\gamma(u^0_1, u^0_2) = \mathcal{E}_\gamma(u^0_1, u^0_2), \qquad (2.2.27)$$

where the Ginzburg-Landau energy functional, $\mathcal{E}_\gamma(\cdot)$ is defined by

$$\mathcal{E}_\gamma(u_1, u_2) = \frac{\gamma}{2}\Big(|u_1|^2_1 + |u_2|^2_1\Big) + (\psi(u_1), 1) + (\psi(u_2), 1) + D((u_1 + 1)(u_2 + 1), 1). \qquad (2.2.28)$$

In particular, since $u_i^0 \in H^1(\Omega)$, we obtain that

$$\mathcal{E}_\gamma^\epsilon(P^k u_1^0, P^k u_2^0) \le C,$$

where C is independent of ϵ and k. Thus by using (2.2.13), (2.2.18d), and (2.2.25) we have

$$\frac{\gamma}{2}|u_{\epsilon,1}^k|_1^2 + \frac{\gamma}{2}|u_{\epsilon,2}^k|_1^2 + \int_0^t |w_{\epsilon,1}^k(s)|_1^2\,ds + \int_0^t |w_{\epsilon,2}^k(s)|_1^2\,ds \le C, \tag{2.2.29}$$

where C is independent of T, ϵ, and k. Noting (2.1.14a), for $i = 1, 2$, to give

$$\begin{aligned}
\left\|\frac{du_{\epsilon,i}^k}{dt}\right\|_{-1}^2 &= \left(\frac{du_{\epsilon,i}^k}{dt}, \mathcal{G}_N \frac{du_{\epsilon,i}^k}{dt}\right) = \left(\frac{du_{\epsilon,i}^k}{dt}, P^k \mathcal{G}_N \frac{du_{\epsilon,i}^k}{dt}\right) \\
&= -\left(\nabla w_{\epsilon,i}^k, \nabla P^k \mathcal{G}_N \frac{du_{\epsilon,i}^k}{dt}\right) = -\left(\nabla w_{\epsilon,i}^k, \nabla \mathcal{G}_N \frac{du_{\epsilon,i}^k}{dt}\right) \\
&= -\left(w_{\epsilon,i}^k, \frac{du_{\epsilon,i}^k}{dt}\right) = |w_{\epsilon,i}^k|_1^2,
\end{aligned} \tag{2.2.30}$$

then setting $t = T$, (2.2.29) can be rewritten as

$$\frac{\gamma}{2}|u_{\epsilon,1}^k(T)|_1^2 + \frac{\gamma}{2}|u_{\epsilon,2}^k(T)|_1^2 + \int_0^T |w_{\epsilon,1}^k(t)|_1^2\,dt + \int_0^T |w_{\epsilon,2}^k(t)|_1^2\,dt \le C, \tag{2.2.31}$$

in particular

$$\int_0^T \left\|\frac{du_{\epsilon,1}^k}{dt}\right\|_{-1}^2 dt + \int_0^T \left\|\frac{du_{\epsilon,2}^k}{dt}\right\|_{-1}^2 dt \le C,$$

which implies, for $i = 1, 2$, that

$$\left\|\frac{du_{\epsilon,i}^k}{dt}\right\|_{L^2(0,T;(H^1(\Omega))')} \le C, \tag{2.2.32}$$

where C is independent of T, ϵ, and k.

Substituting $\eta^k = 1$ in (2.2.19a) yields

$$\left(\frac{\partial u_{\epsilon,i}^k}{\partial t}, 1\right) = 0. \tag{2.2.33}$$

Integrating both side of (2.2.33) over $(0, t)$ to give

$$\begin{aligned}
0 &= \int_0^t \int_\Omega \frac{\partial u_{\epsilon,i}^k}{\partial s}\,dx\,ds \\
&= \int_\Omega \int_0^t \frac{\partial u_{\epsilon,i}^k}{\partial s}\,ds\,dx \\
&= \int_\Omega \left(u_{\epsilon,i}^k(t) - u_{\epsilon,i}^k(0)\right) dx \\
&= (u_{\epsilon,i}^k(t), 1) - (u_{\epsilon,i}^k(0), 1).
\end{aligned}$$

Hence we obtain

$$(u^k_{\epsilon,i}(t),1) \;=\; (u^k_{\epsilon,i}(0),1) \;=\; (P^k(u^0_i),1) \;=\; (u^0_i,1),$$

which implies for any t that

$$|(u^k_{\epsilon,i}(t),1)| \;\leq\; C. \tag{2.2.34}$$

Using the Poincaré inequality (2.1.3), (2.2.29) and (2.2.34), it follows that

$$|u^k_{\epsilon,i}(t)|_0 \;\leq\; C_p\Big(|(u^k_{\epsilon,i}(t),1)| + |u^k_{\epsilon,i}(t)|_1\Big) \;\leq\; C. \tag{2.2.35}$$

The equations (2.2.35) implies that $u^k_{\epsilon,i}(t) \in H^1(\Omega)$. On noting (2.2.35), we obtain the bound

$$\|u^k_{\epsilon,i}\|_{L^\infty(0,T;H^1(\Omega))} \;\leq\; C. \tag{2.2.36}$$

Defining the mean integral by

$$\fint \eta \;:=\; \frac{1}{|\Omega|}(\eta,1) \qquad \forall \eta \in L^2(\Omega),$$

thus we have

$$\begin{aligned}
u^k_{\epsilon,i}(t) - \fint u^k_{\epsilon,i}(t) \;&=\; u^k_{\epsilon,i}(t) - \frac{1}{|\Omega|}(u^k_{\epsilon,i}(t),1)\\
&=\; u^k_{\epsilon,i}(t) - \frac{1}{|\Omega|}(u^k_{\epsilon,i}(0),1)\\
&=\; u^k_{\epsilon,i}(t) - u^k_{\epsilon,i}(0) + u^k_{\epsilon,i}(0) - \frac{1}{|\Omega|}(u^k_{\epsilon,i}(0),1)\\
&=\; \int_0^t \frac{du^k_{\epsilon,i}}{ds}ds + u^k_{\epsilon,i}(0) - \frac{1}{|\Omega|}(u^k_{\epsilon,i}(0),1).
\end{aligned}$$

Noting (2.1.7) and the Young's inequality (2.1.8) yields

$$\begin{aligned}
\left\|u^k_{\epsilon,i}(t) - \fint u^k_{\epsilon,i}(t)\right\|^2_{-1} \;&\leq\; \left(\left\|\int_0^t \frac{du^k_{\epsilon,i}}{ds}ds\right\|_{-1} + \left\|u^k_{\epsilon,i}(0) - \frac{1}{|\Omega|}(u^k_{\epsilon,i}(0),1)\right\|_{-1}\right)^2\\
&\leq\; \left(\left\|\int_0^t \frac{du^k_{\epsilon,i}}{ds}ds\right\|_{-1} + C_p\left|u^k_{\epsilon,i}(0) - \frac{1}{|\Omega|}(u^k_{\epsilon,i}(0),1)\right|_0\right)^2\\
&\leq\; \left(\left\|\int_0^t \frac{du^k_{\epsilon,i}}{ds}ds\right\|_{-1} + C_p|u^k_{\epsilon,i}(0)|_0 + C|(u^k_{\epsilon,i}(0),1)|_0\right)^2\\
&\leq\; C\int_0^t \left\|\frac{du^k_{\epsilon,i}}{ds}\right\|^2_{-1} ds + C\\
&\leq\; C.
\end{aligned} \tag{2.2.37}$$

Integrating (2.2.37) over $(0,T)$ to obtain

$$\left\| u^k_{\epsilon,i}(t) - \fint u^k_{\epsilon,i}(t) \right\|_{L^2(0,T;(H^1(\Omega))')} \le C(T) \le C. \tag{2.2.38}$$

Therefore (2.2.32) and (2.2.38) imply that

$$\|u^k_{\epsilon,i}\|_{H^1(0,T;(H^1(\Omega))')} \le C.$$

From (2.2.31), to show $\int_0^T \|w^k_{\epsilon,i}\|_1 dt$ is bounded, from (2.2.29) and the Poincaré ineqality (2.1.3), it is sufficient to estimate $|(w^k,1)|$. We set $\eta = w^k_{\epsilon,i}$ in the Poincaŕe inequality (2.1.3) and note the Cauchy's inequality (2.1.9) to have

$$\begin{aligned} |w^k_{\epsilon,i}|_0^2 &\le C_p^2\Big(|(w^k_{\epsilon,i},1)| + |w^k_{\epsilon,i}|_1\Big)^2 \\ &\le C\Big(|(w^k_{\epsilon,i},1)|^2 + |w^k_{\epsilon,i}|_1^2\Big). \end{aligned} \tag{2.2.39}$$

Recalling that

$$\|w^k_{\epsilon,i}\|_1^2 = |w^k_{\epsilon,i}|_0^2 + |w^k_{\epsilon,i}|_1^2 \tag{2.2.40}$$

and substituting (2.2.39) into (2.2.40) gives

$$\begin{aligned} \|w^k_{\epsilon,i}\|_1^2 &\le C\Big(|(w^k_{\epsilon,i},1)|^2 + |w^k_{\epsilon,i}|_1^2\Big) + |w^k_{\epsilon,i}|_1^2 \\ &\le C\Big(|(w^k_{\epsilon,i},1)|^2 + |w^k_{\epsilon,i}|_1^2\Big). \end{aligned}$$

Taking $\eta^k = 1$ in (2.2.19b), for $i = 1,2$, we have

$$(w^k_{\epsilon,i},1) = \frac{1}{\epsilon}(\beta_\epsilon(u^k_{\epsilon,i}),1) - (u^k_{\epsilon,i},1) + D(u^k_{\epsilon,j}+1,1) \qquad i \ne j,$$

which implies that

$$|(w^k_{\epsilon,i}(t),1)| \le \left|\frac{1}{\epsilon}(\beta_\epsilon(u^k_{\epsilon,i}),1)\right| + |(u^k_{\epsilon,i},1)| + |D(u^k_{\epsilon,j}+1,1)|.$$

As $\beta_\epsilon \equiv 0$ for $[-1,1]$, it follows from the definition (2.2.6) of β_ϵ that $|\beta_\epsilon(r)| \le r\beta_\epsilon(r)$, hence

$$\frac{1}{\epsilon}(|\beta_\epsilon(u^k_{\epsilon,i}),1)| \le \frac{1}{\epsilon}(\beta_\epsilon(u^k_{\epsilon,i}),u^k_{\epsilon,i}),$$

Thus, we have

$$|(w^k_{\epsilon,i}(t),1)| \le \frac{1}{\epsilon}(\beta_\epsilon(u^k_{\epsilon,i}),u^k_{\epsilon,i}) + |(u^k_{\epsilon,i},1)| + |D(u^k_{\epsilon,j}+1,1)|. \tag{2.2.41}$$

Substituting $\eta^k = u^k_{\epsilon,i}$ in (2.2.19b) to give

$$\begin{aligned}\frac{1}{\epsilon}(\beta_\epsilon(u^k_{\epsilon,i}), u^k_{\epsilon,i}) &= (u^k_{\epsilon,i}, u^k_{\epsilon,i}) - \gamma(\nabla u^k_{\epsilon,i}, \nabla u^k_{\epsilon,i}) + D(u^k_{\epsilon,j}+1, u^k_{\epsilon,i}) + (w^k_{\epsilon,i}, u^k_{\epsilon,i}) \\ &= |u^k_{\epsilon,i}|_0^2 - \gamma|u^k_{\epsilon,i}|_1^2 + D(u^k_{\epsilon,j}+1, u^k_{\epsilon,i}) + (w^k_{\epsilon,i}, u^k_{\epsilon,i}),\end{aligned}$$

hence (2.2.41) becomes

$$|(w^k_{\epsilon,i}, 1)| \le |u^k_{\epsilon,i}|_0^2 - \gamma|u^k_{\epsilon,i}|_1^2 + D(u^k_{\epsilon,j}+1, u^k_{\epsilon,i}) + (w^k_{\epsilon,i}, u^k_{\epsilon,i}) + |(u^k_{\epsilon,i}, 1)| + |D(u^k_{\epsilon,j}+1, 1)|. \tag{2.2.42}$$

Now using (2.1.1a) and noting $(u^k_{\epsilon,i}, 1) = m_i$ yield

$$\begin{aligned}(w^k_{\epsilon,i}, u^k_{\epsilon,i}) &= \left(w^k_{\epsilon,i}, u^k_{\epsilon,i} - \frac{1}{|\Omega|}(u^k_{\epsilon,i}, 1)\right) + \frac{1}{|\Omega|}(u^k_{\epsilon,i}, 1)(w^k_{\epsilon,i}, 1) \\ &= \left(\nabla w^k_{\epsilon,i}, \nabla \mathcal{G}_N\left(u^k_{\epsilon,i} - \frac{1}{|\Omega|}(u^k_{\epsilon,i}, 1)\right)\right) + \frac{m_i}{|\Omega|}(w^k_{\epsilon,i}, 1) \\ &\le |w^k_{\epsilon,i}|_1 \left|\mathcal{G}_N\left(u^k_{\epsilon,i} - \frac{m_i}{|\Omega|}\right)\right|_1 + \frac{m_i}{|\Omega|}(w^k_{\epsilon,i}, 1).\end{aligned} \tag{2.2.43}$$

As $|(u^k_{\epsilon,i}, 1)| = |m_i| < |\Omega|$, rearranging (2.2.42) and using (2.2.43) and (2.1.7), we obtain

$$\begin{aligned}&|(w^k_{\epsilon,i}, 1)| \\ &\le |u^k_{\epsilon,i}|_0^2 - \gamma|u^k_{\epsilon,i}|_1^2 + D(u^k_{\epsilon,j}+1, u^k_{\epsilon,i}) + |m_i| + D|m_j + |\Omega|| + |w^k_{\epsilon,i}|_1 \left\|u^k_{\epsilon,i} - \frac{m_i}{|\Omega|}\right\|_{-1} \\ &\quad + \frac{|m_i|}{|\Omega|}|(w^k_{\epsilon,i}, 1)| \\ &\le |u^k_{\epsilon,i}|_0^2 - \gamma|u^k_{\epsilon,i}|_1^2 + D(u^k_{\epsilon,j}+1, u^k_{\epsilon,i}) + |m_i| + D|m_j + |\Omega|| + C_p|w^k_{\epsilon,i}|_1 \left|u^k_{\epsilon,i} - \frac{m_i}{|\Omega|}\right|_0 \\ &\quad + \frac{|m_i|}{|\Omega|}|(w^k_{\epsilon,i}, 1)| \\ &\le \frac{|m_i| + |u^k_{\epsilon,i}|_0^2 - \gamma|u^k_{\epsilon,i}|_1^2 + C_p|w^k_{\epsilon,i}|_1 \left|u^k_{\epsilon,i} - \frac{m_i}{|\Omega|}\right|_0 + D(u^k_{\epsilon,j}+1, u^k_{\epsilon,i}) + D|m_j + |\Omega||}{1 - \frac{|m_i|}{|\Omega|}}.\end{aligned} \tag{2.2.44}$$

In order to bound the terms on the right-hand side of (2.2.44), we use the Cauchy's inequality (2.1.9) and (2.2.36) to have

$$\begin{aligned}D((u^k_{\epsilon,j}+1), u^k_{\epsilon,i}) &= D\Big((u^k_{\epsilon,j}, u^k_{\epsilon,i}) + m_i\Big) \\ &\le D\left(\frac{1}{2}|u^k_{\epsilon,j}|_0^2 + \frac{1}{2}|u^k_{\epsilon,i}|_0^2 + m_i\right) \\ &\le C,\end{aligned} \tag{2.2.45}$$

and

$$|D((u^k_{\epsilon,j}+1),1)| = \left|D\int_\Omega (u^k_{\epsilon,j}+1)dx\right| \leq C. \tag{2.2.46}$$

Noting (2.2.44), (2.2.45), and (2.2.46) we conclude that

$$|(w^k_{\epsilon,i},1)| \leq C\Big(|w^k|_1+1\Big). \tag{2.2.47}$$

Noting (2.2.29), then integrating (2.2.40) over $(0,T)$ and noting (2.2.29) yield

$$\|w^k_{\epsilon,i}\|_{L^2(0,T;H^1(\Omega))} \leq C\Big(1+|\nabla w^k_{\epsilon,i}|_{L^2(\Omega_T)}\Big) \leq C\Big(1+T^{\frac{1}{2}}\Big).$$

Furthermore, since $L^\infty(0,T;H^1(\Omega)) \subset L^2(0,T;H^1(\Omega))$, it follows that

$$\|u^k_{\epsilon,i}\|_{L^2(0,T;H^1(\Omega))} \leq C.$$

Thus we arrive at $u^k_{\epsilon,i} \in H^1(0,T;(H^1(\Omega))')\bigcap L^2(0,T;H^1(\Omega))$.

From compactness arguments we deduce the existence of subsequences $(u^k_{\epsilon,i}, w^k_{\epsilon,i})$ having the following properties:

$$u^k_{\epsilon,i} \rightharpoonup u_{\epsilon,i} \text{ in } H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^1(\Omega)), \tag{2.2.48a}$$

$$u^k_{\epsilon,i} \overset{*}{\rightharpoonup} u_{\epsilon,i} \text{ in } L^\infty(0,T;H^1(\Omega)), \tag{2.2.48b}$$

$$w^k_{\epsilon,i} \rightharpoonup w_{\epsilon,i} \text{ in } L^2(0,T;H^1(\Omega)), \tag{2.2.48c}$$

$$u^k_{\epsilon,i} \to u_{\epsilon,i} \text{ in } L^2(\Omega_T), \tag{2.2.48d}$$

(2.2.48d) being a consequence of a compactness theorem of Lions, see Appendix A.

Furthermore, as the compact embedding $H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^1(\Omega)) \hookrightarrow C([0,T];L^2(\Omega))$ which together with (2.2.48a) and the strong convergence in $L^2(\Omega)$ of $P^k(u^0_i)$ to u^0_i implies that $u_{\epsilon,i}(0) = u^0_i$.

Now we pass to the limit in k. For any $\eta \in H^1(\Omega)$, set $\eta^k = P^k\eta$ in (2.2.19a - b) to obtain

$$\left(\frac{\partial u^k_{\epsilon,i}}{\partial t}, P^k\eta\right) + (\nabla w^k_{\epsilon,i}, \nabla P^k\eta) = 0, \quad \forall \eta \in H^1(\Omega), \tag{2.2.49a}$$

$$\gamma(\nabla u^k_{\epsilon,i}, \nabla P^k\eta) + \frac{1}{\epsilon}(\beta_\epsilon(u^k_{\epsilon,i}), P^k\eta) - (u^k_{\epsilon,i}, P^k\eta) + D(u^k_{\epsilon,j}+1, P^k\eta) = (w^k_{\epsilon,i}, P^k\eta). \tag{2.2.49b}$$

We can immediately pass to the limit *a.e.* in (2.2.49a) to obtain (2.2.17a). To yield the result it remains to prove that for $i = 1, 2$,

$$(\beta_\epsilon(u^k_{\epsilon,i}), P^k\eta) \to (\beta_\epsilon(u_{\epsilon,i}), \eta) \qquad \text{as } k \to \infty.$$

This is proven by simply using the Lipschitz continuity of β_ϵ, properties of P^k and the strong convegence of $u^k_{\epsilon,i}$ to $u_{\epsilon,i}$ in $L^2(\Omega)$, for *a.e* $t \in (0, T)$ to have that

$$\begin{aligned}|(\beta_\epsilon(u^k_{\epsilon,i}), P^k\eta) - (\beta_\epsilon(u_{\epsilon,i}), \eta)| &\leq |(\beta_\epsilon(u^k_{\epsilon,i}) - \beta_\epsilon(u_{\epsilon,i}), P^k\eta) + (\beta_\epsilon(u_{\epsilon,i}), P^k\eta - \eta)|, \\ &\leq |u^k_{\epsilon,i} - u_{\epsilon,i}|_0|\eta|_0 + |\beta_\epsilon(u_{\epsilon,i})|_0|\eta - P^k\eta|_0 \to 0 \text{ as } k \to \infty\end{aligned}$$

hence we obtain

$$\gamma(\nabla u_{\epsilon,i}, \nabla\eta) + (\psi'_\epsilon(u_{\epsilon,i}), \eta) + D(u_{\epsilon,j} + 1, \eta) = (w_{\epsilon,i}, \eta).$$

Finally we prove uniqueness. Let $\{u_{\epsilon,1}, u_{\epsilon,2}, w_{\epsilon,1}, w_{\epsilon,2}\}$ and $\{u^*_{\epsilon,1}, u^*_{\epsilon,2}, w^*_{\epsilon,1}, w^*_{\epsilon,2}\}$ be two different solutions to Problem $(\mathbf{P}_\epsilon)$, define, for $i = 1, 2$, $\theta^u_i = u_{\epsilon,i} - u^*_{\epsilon,i}$ and $\theta^w_i = w_{\epsilon,i} - w^*_{\epsilon,i}$. Substract (2.2.17a), when $\{u_{\epsilon,1}, u_{\epsilon,2}, w_{\epsilon,1}, w_{\epsilon,2}\}$ is the solution, from (2.2.17a), when $\{u^*_{\epsilon,1}, u^*_{\epsilon,2}, w^*_{\epsilon,1}, w^*_{\epsilon,2}\}$ is the solution

$$\begin{aligned}\left\langle \frac{\partial(u_{\epsilon,i} - u^*_{\epsilon,i})}{\partial t}, \eta \right\rangle + (\nabla(w_{\epsilon,i} - w^*_{\epsilon,i}), \nabla\eta) &= 0 \quad \forall\eta \in H^1(\Omega), \\ \text{then} \quad \left\langle \frac{\partial\theta^u_i}{\partial t}, \eta \right\rangle + (\nabla\theta^w_i, \nabla\eta) &= 0 \quad \forall\eta \in H^1(\Omega), \end{aligned} \tag{2.2.50}$$

and subtract (2.2.17b), when $\{u_{\epsilon,1}, u_{\epsilon,2}, w_{\epsilon,1}, w_{\epsilon,2}\}$ is the solution and $\eta = \theta^u_i$, from (2.2.17b), when $\{u^*_{\epsilon,1}, u^*_{\epsilon,2}, w^*_{\epsilon,1}, w^*_{\epsilon,2}\}$ is the solution and $\eta = \theta^u_i$. Then use the monotonicity of ψ'_ϵ and (2.2.10) yields, for $j = 1, 2$, $i \neq j$,

$$\gamma(\nabla(u_{\epsilon,i} - u^*_{\epsilon,i}), \nabla\theta^u_i) + (\psi'_\epsilon(u_{\epsilon,i}) - \psi'_\epsilon(u^*_{\epsilon,i}), \theta^u_i) + D(u_{\epsilon,j} - u^*_{\epsilon,j}, \theta^u_i) = (w_{\epsilon,i} - w^*_{\epsilon,i}, \theta^u_i),$$

$$\text{so that} \quad \gamma|\theta^u_i|^2_1 - |\theta^u_i|^2_0 + D(\theta^u_j, \theta^u_i) \leq (\theta^w_i, \theta^u_i). \tag{2.2.51}$$

Setting $\eta = \mathcal{G}_N\theta^u_i$ in (2.2.50), noting (2.1.1a) and (2.1.2) to obtain

$$(\theta^w_i, \theta^u_i) = -\left\langle \frac{\partial\theta^u_i}{\partial t}, \mathcal{G}_N\theta^u_i \right\rangle, \tag{2.2.52}$$

and then substituting (2.2.52) into (2.2.51) yields

$$\left\langle \frac{\partial\theta^u_1}{\partial t}, \mathcal{G}_N\theta^u_1 \right\rangle + \gamma|\theta^u_1|^2_1 + D(\theta^u_2, \theta^u_1) \leq |\theta^u_1|^2_0, \tag{2.2.53}$$

$$\left\langle \frac{\partial \theta_2^u}{\partial t}, \mathcal{G}_N \theta_2^u \right\rangle + \gamma|\theta_2^u|_1^2 + D(\theta_1^u, \theta_2^u) \ \leq \ |\theta_2^u|_0^2. \tag{2.2.54}$$

Now using the definition of $\mathcal{G}_N$ (2.1.1a,b) and (2.1.4), we then have the identity

$$\left\langle \frac{\partial \theta_1^u}{\partial t}, \mathcal{G}_N \theta_1^u \right\rangle = \frac{1}{2}\frac{d}{dt}\|\theta_1^u\|_{-1}^2,$$

$$\left\langle \frac{\partial \theta_2^u}{\partial t}, \mathcal{G}_N \theta_2^u \right\rangle = \frac{1}{2}\frac{d}{dt}\|\theta_2^u\|_{-1}^2.$$

Adding (2.2.53) to (2.2.54) gives

$$\frac{1}{2}\left(\frac{d}{dt}\|\theta_1^u\|_{-1}^2 + \frac{d}{dt}\|\theta_2^u\|_{-1}^2\right) + \gamma\Big(|\theta_1^u|_1^2 + \theta_2^u|_1^2\Big) + 2D(\theta_1^u, \theta_2^u) \ \leq \ |\theta_1^u|_0^2 + |\theta_2^u|_0^2.$$

Noting the Cauchy-Schwarz inequality, the Cauchy's inequality (2.1.9) and (2.1.10), we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\theta_1^u\|_{-1}^2 + \theta_2^u\|_{-1}^2\Big) + \gamma\Big(|\theta_1^u|_1^2 + \theta_2^u|_1^2\Big) \ &\leq \ (1+D)\Big(|\theta_1^u|_0^2 + |\theta_2^u|_0^2\Big) \\ &\leq \ \frac{\gamma}{2}\Big(|\theta_1^u|_1^2 + |\theta_2^u|_1^2\Big) \\ &\quad + \frac{(1+D)^2}{2\gamma}\Big(\|\theta_1^u\|_{-1}^2 + \|\theta_2^u\|_{-1}^2\Big),\end{aligned} \tag{2.2.55}$$

so that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\theta_1^u\|_{-1}^2 + \|\theta_2^u\|_{-1}^2\Big) + \frac{\gamma}{2}\Big(|\theta_1^u|_1^2 + |\theta_2^u|_1^2\Big) \ &\leq \ \frac{(1+D)^2}{2\gamma}\Big(\|\theta_1^u\|_{-1}^2 + \|\theta_2^u\|_{-1}^2\Big), \\ \frac{d}{dt}\Big(\|\theta_1^u\|_{-1}^2 + \theta_2^u\|_{-1}^2\Big) + \gamma\Big(|\theta_1^u|_1^2 + \theta_2^u|_1^2\Big) \ &\leq \ \frac{C}{\gamma}\Big(\|\theta_1^u\|_{-1}^2 + \|\theta_2^u\|_{-1}^2\Big),\end{aligned}$$

then multiplying through by $\exp\left(\frac{-Ct}{\gamma}\right)$ and integrating over $(0,t)$ yield

$$\begin{aligned}\exp\left(\frac{-Ct}{\gamma}\right)\Big(\|\theta_1^u(t)\|_{-1}^2 + \|\theta_2^u(t)\|_{-1}^2\Big) + \gamma\int_0^t \exp\left(\frac{-Cs}{\gamma}\right)\Big(|\theta_1^u(s)|_1^2 + |\theta_2^u(s)|_1^2\Big)ds \\ \leq \ \|\theta_1^u(0)\|_{-1}^2 + \|\theta_2^u(0)\|_{-1}^2 \ \leq 0.\end{aligned} \tag{2.2.56}$$

Also noting that from (2.2.17a), $(\theta_i^u, 1) = 0$ and noting the Poincaré inequality (2.1.3), we obtain the uniqueness of $u_{\epsilon,i}$. Now using (2.2.17b) we also obtain the uniqueness of $w_{\epsilon,i}$, thus proving existence and uniqueness to the Problem $(\mathbf{P}_\epsilon)$.

Setting $s = u_{\epsilon,i}$ and $r = u^k_{\epsilon,i}$ in (2.2.12), and using (2.2.18d) we have

$$\begin{aligned}
&\mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}, u^k_{\epsilon,2}) \\
&= \frac{\gamma}{2}\left(|u^k_{\epsilon,1}|^2_1 + |u^k_{\epsilon,2}|^2_1\right) + (\psi_\epsilon(u^k_{\epsilon,1}), 1) + (\psi_\epsilon(u^k_{\epsilon,2}), 1) + D((u^k_{\epsilon,1}+1)(u^k_{\epsilon,2}+1), 1) \\
&\geq \mathcal{E}_\gamma(u^k_{\epsilon,1}, u^k_{\epsilon,2}) \\
&\quad + (\psi_{\epsilon,1}(u_{\epsilon,1}), 1) + (\psi_{\epsilon,1}(u_{\epsilon,2}), 1) + \frac{1}{\epsilon}(\beta_\epsilon(u_{\epsilon,1}), u^k_{\epsilon,1} - u_{\epsilon,1}) + \frac{1}{\epsilon}(\beta_\epsilon(u_{\epsilon,2}), u^k_{\epsilon,2} - u_{\epsilon,2}),
\end{aligned}$$

hence noting the convergence of $u^k_{\epsilon,i}$ to $u_{\epsilon,i}$ in $H^1(\Omega)$ gives

$$\liminf_{k\to\infty} \mathcal{E}^\epsilon_\gamma(u^k_{\epsilon,1}, u^k_{\epsilon,2}) \geq \mathcal{E}_\gamma(u_{\epsilon,1}, u_{\epsilon,2}) + (\psi_{\epsilon,1}(u_{\epsilon,1}), 1) + (\psi_{\epsilon,1}(u_{\epsilon,2}), 1) \;=\; \mathcal{E}^\epsilon_\gamma(u_{\epsilon,1}, u_{\epsilon,2})$$

which together with (2.2.25) and (2.2.27) yields that

$$\mathcal{E}^\epsilon_\gamma(u_{\epsilon,1}, u_{\epsilon,2}) + \int_0^t |w_{\epsilon,1}(s)|^2_1 ds + \int_0^t |w_{\epsilon,2}(s)|^2_1 ds \;\leq\; \mathcal{E}^\epsilon_\gamma(u^0_1, u^0_2). \tag{2.2.57}$$

Using a "stop start" argument, as $u_{\epsilon,i} \in C([\delta, T]; H^1(\Omega)), \forall \delta > 0$ for $t' \geq 0$ we may set $U^0_i = u_{\epsilon,i}(t')$ and then solve Problem $(\mathbf{P}_\epsilon)$ with the initial data U^0_i to obtain $U_i(t)\,\forall t \geq t'$ which satisfies

$$\mathcal{E}^\epsilon_\gamma(U_1(t), U_2(t)) + \int_{t'}^t \|U_{1,t}(s)\|^2_{-1} ds + \int_{t'}^t \|U_{2,t}\|^2_{-1} ds \;\leq\; \mathcal{E}^\epsilon_\gamma(u_{\epsilon,1}(t'), u_{\epsilon,2}(t')).$$

By uniqueness we have that $u_{\epsilon,i}(t) \equiv U_i(t)$ and so we have a Lyapunov functional.
□

In order to prove existence and uniqueness of Problem $(\mathbf{P})$ and Problem $(\mathbf{Q})$, we define $\beta : \mathbb{R} \to \mathbb{R}$ as follows:

$$\beta(r) \;=\; \lim_{\epsilon\to 0} \beta_\epsilon(r) \;=\; \begin{cases} r-1 & \text{for } r \geq 1, \\ 0 & \text{for } |r| \leq 1, \\ r+1 & \text{for } r \leq -1. \end{cases}$$

We note that β is a Lipschitz continuous function,

$$|\beta(r) - \beta_\epsilon(r)| \;\leq\; \frac{\epsilon}{2} \quad \forall r \in \mathbb{R}, \text{ and } |\beta(r) - \beta(s)| \leq |r - s| \;\forall r, s \in \mathbb{R}. \tag{2.2.58}$$

Theorem 2.2.3 There exists unique solutions to Problem (**P**) and Problem (**Q**) such that, for $i,j=1,2$, $i\neq j$,

$$\|u_i\|_{H^1(0,T;(H^1(\Omega))')} \le C, \tag{2.2.59a}$$

$$\|u_i\|_{L^\infty(0,T;H^1(\Omega))} \le C, \tag{2.2.59b}$$

$$\|w_i\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{2.2.59c}$$

where C is independent of T.

The Ginzburg-Landau energy functional, $\mathcal{E}_\gamma(\cdot)$ defined as (2.2.28) is the Lyapunov functional for Problem (**P**) and Problem (**Q**). Also given initial data u_0 and v_0 and denoting the solutions to Problem (**Q**) by $u(t)$ and $v(t)$ respectively then

$$\|u_1(t)-v_1(t)\|_{-1}+\|u_2(t)-v_2(t)\|_{-1} \le C(t)\Big(\|u_1^0-v_1^0\|_{-1}+\|u_2^0-v_2^0\|_{-1}\Big). \tag{2.2.59d}$$

Proof: We observe that $u_{\epsilon,i}\in L^\infty(0,T;H^1(\Omega))\cap H^1(0,T;(H^1(\Omega))')$ and $w_{\epsilon,i}\in L^2(0,T;H^1(\Omega))$ are bounded independently of ϵ, thus by the compactness theorem, see Appendix A, there exists a subsequence of $\{u_{\epsilon,i}\}$ and $\{w_{\epsilon,i}\}$ such that

$$u_{\epsilon,i} \rightharpoonup u_i \text{ in } H^1(0,T;(H^1(\Omega))')\cap L^2(0,T;H^1(\Omega)), \tag{2.2.60a}$$

$$u_{\epsilon,i} \overset{*}{\rightharpoonup} u_i \text{ in } L^\infty(0,T;H^1(\Omega)), \tag{2.2.60b}$$

$$w_{\epsilon,i} \rightharpoonup w_i \text{ in } L^2(0,T;H^1(\Omega)), \tag{2.2.60c}$$

$$u_{\epsilon,i} \to u_i \text{ in } L^2(\Omega_T). \tag{2.2.60d}$$

Therefore (2.2.59a-c) follow on noting (2.2.60a-c).

We now consider

$$\begin{aligned}\left|\lim_{\epsilon\to0}\|u_{\epsilon,i}\|_0^2-\|u_i\|_0^2\right| &= \lim_{\epsilon\to0}\left|\|u_{\epsilon,i}\|_0^2-\|u_i\|_0^2\right| = \lim_{\epsilon\to0}|(u_{\epsilon,i}-u_i,u_{\epsilon,i}+u_i)| \\ &\le \lim_{\epsilon\to0}\|u_{\epsilon,i}-u_i\|_0\Big(\|u_{\epsilon,i}\|_0+\|u_i\|_0\Big).\end{aligned} \tag{2.2.61}$$

Then noting (2.2.60d) and the convergence properties, see Appendix A, we have

$$\lim_{\epsilon\to0}|u_{\epsilon,i}|_0^2 = |u_i|_0^2. \tag{2.2.62}$$

Furthermore, from the convergence properties in (2.2.60a-c), we can pass to the limit in (2.2.17a) to obtain (2.2.1a). Now, setting $\eta = \beta_\epsilon(u_{\epsilon,i}) \in H^1(\Omega)$ in (2.2.17b), and using the Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\gamma(\nabla u_{\epsilon,i}, \nabla\beta_\epsilon(u_{\epsilon,i})) + \frac{1}{\epsilon}|\beta_\epsilon(u_{\epsilon,i})|_0^2 \\
&= (u_{\epsilon,i} + w_{\epsilon,i}, \beta_\epsilon(u_{\epsilon,i})) - D(u_j^\epsilon + 1, \beta_\epsilon(u_{\epsilon,i})) \\
&\leq \Big(|u_{\epsilon,i}|_0 + |w_{\epsilon,i}|_0 + D|u_j^\epsilon + 1|_0\Big)|\beta_\epsilon(u_{\epsilon,i})|_0 \\
&\leq \epsilon\Big(|u_{\epsilon,i}|_0^2 + |w_{\epsilon,i}|_0^2\Big) + \frac{1}{2\epsilon}|\beta_\epsilon(u_{\epsilon,i})|_0^2 + D^2\epsilon|u_{\epsilon,j} + 1|_0^2 + \frac{1}{4\epsilon}|\beta_\epsilon(u_{\epsilon,i})|_0^2 \\
&= \epsilon\Big(|u_{\epsilon,i}|_0^2 + |w_{\epsilon,i}|_0^2\Big) + \frac{3}{4\epsilon}|\beta_\epsilon(u_{\epsilon,i})|_0^2 + D^2\epsilon|u_{\epsilon,j} + 1|_0^2, \qquad (2.2.63)
\end{aligned}$$

and, as $0 \leq \beta'_\epsilon \leq 1$, we also obtain

$$\begin{aligned}
(\nabla u_{\epsilon,i}, \nabla\beta_\epsilon(u_{\epsilon,i})) &= \int_\Omega \beta'_\epsilon(u_{\epsilon,i})\nabla u_{\epsilon,i}\cdot\nabla u_{\epsilon,i}dx \\
&\geq \int_\Omega (\beta'_\epsilon(u_{\epsilon,i}))^2\nabla u_{\epsilon,i}\cdot\nabla u_{\epsilon,i}dx \\
&= |\beta_\epsilon(u_{\epsilon,i})|_1^2 \geq 0, \qquad (2.2.64)
\end{aligned}$$

it follows from (2.2.63) that

$$\|\beta_\epsilon(u_{\epsilon,i})\|_{L^2(\Omega_T)} \leq C\epsilon. \qquad (2.2.65)$$

Hence, from (2.2.63) and (2.2.64), we note that

$$\|\beta_\epsilon(u_{\epsilon,i})\|_{L^2(0,T;H^1(\Omega))} \leq C\epsilon^{\frac{1}{2}}.$$

If we let $\epsilon \to 0$, then, from (2.2.65), for $a.e.\, t \in (0,T)$,

$$\lim_{\epsilon\to 0}|\beta_\epsilon(u_{\epsilon,i})|_0 = 0.$$

Using the Lipschitz continuity of β (2.2.58) and (2.2.65) gives

$$\begin{aligned}
\int_0^T |(\beta(u_i),\eta)|\,dt &\leq \int_0^T \Big(|\beta(u_i) - \beta(u_{\epsilon,i})|_0 + |\beta(u_{\epsilon,i}) - \beta_\epsilon(u_{\epsilon,i})|_0 + |\beta_\epsilon(u_{\epsilon,i})|_0\Big)|\eta|_0\,dt, \\
&\leq C\Big(|u_i - u_{\epsilon,i}|_{L^2(\Omega_T)} + \epsilon\Big)|\eta|_{L^2(\Omega_T)},
\end{aligned}$$

so that from the compactness theorem of Lions, see Appendix A, $u_{\epsilon,i}$ converges strongly to u_i in $L^2(\Omega_T)$, $\beta(u_i) = 0$ $a.e.$ that is $u_i \in K$.

Let $v \in K$ thus $\beta_\epsilon(v) = 0$ and

$$\begin{aligned}\gamma(\nabla u_{\epsilon,i}, \nabla v - \nabla u_{\epsilon,i}) - (u_{\epsilon,i} + w_{\epsilon,i}, v - u_{\epsilon,i}) + D(u_{\epsilon,j} + 1, v - u_{\epsilon,i})& \\ = \frac{1}{\epsilon}(\beta_\epsilon(v) - \beta_\epsilon(u_{\epsilon,i}), v - u_{\epsilon,i}) &\geq 0\end{aligned}$$

hence we obtain

$$\gamma(\nabla u_{\epsilon,i}, \nabla v - \nabla u_{\epsilon,i}) - (u_{\epsilon,i}, v - u_{\epsilon,i}) + D(u_{\epsilon,j} + 1, v - u_{\epsilon,i}) \geq (w_{\epsilon,i}, v - u_{\epsilon,i}). \quad (2.2.66)$$

Rearranging (2.2.66) to have

$$\begin{aligned}&\gamma(\nabla u_{\epsilon,i}, \nabla v) - (u_{\epsilon,i}, v) + D(u_{\epsilon,j} + 1, v) - (w_{\epsilon,i}, v) \\ &\geq \gamma(\nabla u_{\epsilon,i}, \nabla u_{\epsilon,i}) - (u_{\epsilon,i}, u_{\epsilon,i}) + D(u_{\epsilon,j} + 1, u_{\epsilon,i}) - (w_{\epsilon,i}, u_{\epsilon,i}).\end{aligned}$$

Noting the convergence properties of $u_{\epsilon,i}$ and $w_{\epsilon,i}$, it follows that

$$\begin{aligned}&\gamma(\nabla u_i, \nabla v) - (u_i, v) + D(u_j + 1, v) - (w_i, v) \\ &\geq \gamma(\nabla u_i, \nabla u_i) - (u_i, u_i) + D(u_j + 1, u_i) - (w_i, u_i). \quad (2.2.67)\end{aligned}$$

Rearranging (2.2.67) yields (2.2.1b). Therefore we have proven existence to Problem (**P**).

To prove existence to Problem (**Q**), substituting (2.2.2c) into (2.2.1b), and then restricting $\eta \in K$ to have a fixed mass, *i.e.* $(\eta, 1) = (u_i^0, 1)$ yields existence.

We now prove uniqueness. Let $\{u_1, u_2, w_1, w_2\}$ and $\{u_1^*, u_2^*, w_1^*, w_2^*\}$ be two different solutions to Problem (**P**). For $i, j = 1, 2, i \neq j$, subtracting (2.2.1a) when $\{u_1, u_2, w_1, w_2\}$ is the solution from (2.2.1a) when $\{u_1^*, u_2^*, w_1^*, w_2^*\}$ is the solution gives

$$\left\langle \frac{\partial \theta_i^u}{\partial t}, \eta \right\rangle + (\nabla \theta_i^w, \nabla \theta_i^u) = 0.$$

Repeating the arguments used in proving uniqueness for $u_{\epsilon,i}$ in Problem ($\mathbf{P}_\epsilon$), we prove uniqueness for u_i.

If we note that

$$|\theta_i^w|_1^2 = -\left\langle \frac{\partial \theta_i^u}{\partial t}, \theta_i^w \right\rangle = 0,$$

then we see that w_i is unique up to addition of a constant. As $u_i \in C((0,T];H^1(\Omega))$ we may define in the H^1 sense, the open set

$$\Omega_0(t) \;:=\; \{x \in \Omega : |u_i(x)| < 1\}$$

since $(u_i, 1) = m_i \in (-|\Omega|, |\Omega|)$, $\Omega_0(t)$ is non-empty. Taking $\eta = u_i \pm \delta\phi$ in (2.2.1b) where $\phi \in C_0^\infty(\Omega_0(t))$ and δ is chosen so that $\eta \in K$ then we have

$$\gamma(\nabla u_i, \nabla\phi) + D(u_j + 1, \phi) \;=\; (u_i + w_i, \phi) \quad \forall\phi \in C_0^\infty(\Omega_0(t)), a.e.\, t,$$

we conclude that

$$(\theta_i^w(t), \phi) \;=\; 0 \quad \forall\phi \in C_0^\infty(\Omega_0(t)), \quad a.e.\, t, \tag{2.2.68}$$

from which uniqueness for w_i follows.

Noting the weak convergence of $u_{\epsilon,i}$ to u_i in $H^1(\Omega)$, (2.2.11), (2.2.65), (2.2.18d) and (2.2.28) yields

$$\begin{aligned}
&\liminf_{\epsilon\to 0} \mathcal{E}_\gamma^\epsilon(u_{\epsilon,1}, u_{\epsilon,2}) \\
&= \liminf_{\epsilon\to 0} \mathcal{E}_\gamma(u_{\epsilon,1}, u_{\epsilon,2}) + \liminf_{\epsilon\to 0}(\psi_{\epsilon,1}(u_{\epsilon,1}), 1) + \liminf_{\epsilon\to 0}(\psi_{\epsilon,1}(u_{\epsilon,2}), 1), \\
&\geq \liminf_{\epsilon\to 0} \mathcal{E}_\gamma(u_{\epsilon,1}, u_{\epsilon,2}) + \lim_{\epsilon\to 0} \frac{1}{2\epsilon}|\beta_\epsilon(u_{\epsilon,1})|_0^2 + \liminf_{\epsilon\to 0} \frac{1}{2\epsilon}|\beta_\epsilon(u_{\epsilon,2})|_0^2 \;\geq\; \mathcal{E}_\gamma(u_1, u_2),
\end{aligned}$$

so that by the weak convergence of $w_{\epsilon,i}$ to w_i in $H^1(\Omega)$ and noting (2.2.57), (2.2.1c) and (2.2.4)

$$\mathcal{E}_\gamma(u_1, u_2) + \int_0^t |w_1(s)|_1^2 ds + \int_0^t |w_1(s)|_1^2 ds \;\leq\; \mathcal{E}_\gamma(u_1^0, u_2^0),$$

which proves that $\mathcal{E}_\gamma$ is a Lyapunov functional.

Finally, we prove continuous dependence on the data. Now taking initial data u_1^0, u_2^0, v_1^0, and v_2^0, then, as in the uniqueness proof for Problem $(\mathbf{P}_\epsilon)$, comparing with (2.2.56), it is possible to show that for all $t > 0$

$$\begin{aligned}
&\exp\left(-\frac{ct}{\gamma}\right)\left(\|u_1(t) - v_1(t)\|_{-1}^2 + \|u_2(t) - v_2(t)\|_{-1}^2\right) \\
&+ \gamma\int_0^t \exp\left(-\frac{ct}{\gamma}\right)\left(|u_1(s) - v_1(s)|_1^2 + |u_2(s) - v_2(s)|_1^2\right) ds \\
&\leq \|u_1^0(t) - v_1^0(t)\|_{-1} + \|u_2^0(t) - v_2^0(t)\|_{-1}^2
\end{aligned}$$

from which it is easy to deduce (2.2.59d). □

2.3 Regularity

Theorem 2.3.1 For $i = 1, 2$, let $u_i^0 \in H^3(\Omega)$, $\frac{\partial u_i^0}{\partial \eta} = 0$ on $\partial\Omega$ and $\delta \in (0, 1)$ be such that $\|u_i^0\|_0 \leq 1 - \delta$. Let $d \leq 3$ with either Ω being a convex polyhedron or $\partial\Omega \in C^{1,1}$. Then, for all $\epsilon \leq \epsilon_0(\delta)$, the solution $\{u_{\epsilon,1}, u_{\epsilon,2}, w_{\epsilon,1}, w_{\epsilon,2}\}$ of Problem $(\mathbf{P}_\epsilon)$ is such that the following stability bounds hold independently of ϵ :

$$\left\| \frac{\partial u_{\epsilon,i}}{\partial t} \right\|_{L^2(0,T;H^1(\Omega))} + \left\| \frac{\partial u_{\epsilon,i}}{\partial t} \right\|_{L^\infty(0,T;(H^1(\Omega))')} + \|w_{\epsilon,i}\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \tag{2.3.1a}$$

$$\|u_{\epsilon,i}\|_{L^\infty(0,T;H^2(\Omega))} + \|w_{\epsilon,i}\|_{L^2(0,T;H^2(\Omega))} \leq C. \tag{2.3.1b}$$

Proof: Differentiating (2.2.19b) with respect to t then setting $\eta = \frac{\partial u_{\epsilon,i}^k}{\partial t}$, for $i, j = 1, 2$, $i \neq j$, and setting $\eta = \frac{\partial w_{\epsilon,i}^k}{\partial t}$ in (2.2.19a) give

$$\begin{aligned} \gamma \left| \frac{\partial u_{\epsilon,i}^k}{\partial t} \right|_1^2 + \frac{1}{\epsilon} \left(\beta_\epsilon'(u_{\epsilon,i}^k) \frac{\partial u_{\epsilon,i}^k}{\partial t}, \frac{\partial u_{\epsilon,i}^k}{\partial t} \right) - \left| \frac{\partial u_{\epsilon,i}^k}{\partial t} \right|_0^2 + D \left(\frac{\partial u_{\epsilon,j}^k}{\partial t}, \frac{\partial u_{\epsilon,i}^k}{\partial t} \right) &= \left(\frac{\partial w_{\epsilon,i}^k}{\partial t}, \frac{\partial u_{\epsilon,i}^k}{\partial t} \right) \\ &= - \left(\nabla w_{\epsilon,i}^k, \nabla \frac{\partial w_{\epsilon,i}^k}{\partial t} \right). \end{aligned}$$

Noting that $\beta_\epsilon'(\cdot) \geq 0$, we obtain

$$\gamma \left| \frac{\partial u_{\epsilon,i}^k}{\partial t} \right|_1^2 + \frac{1}{2} \frac{d}{dt} |w_{\epsilon,i}^k|_1^2 \leq \left| \frac{\partial u_{\epsilon,i}^k}{\partial t} \right|_0^2 - D \left(\frac{\partial u_{\epsilon,j}^k}{\partial t}, \frac{\partial u_{\epsilon,i}^k}{\partial t} \right),$$

hence we have that

$$\gamma \left| \frac{\partial u_{\epsilon,1}^k}{\partial t} \right|_1^2 + \frac{1}{2} \frac{d}{dt} |w_{\epsilon,1}^k|_1^2 \leq \left| \frac{\partial u_{\epsilon,1}^k}{\partial t} \right|_0^2 - D \left(\frac{\partial u_{\epsilon,2}^k}{\partial t}, \frac{\partial u_{\epsilon,1}^k}{\partial t} \right), \tag{2.3.2}$$

and

$$\gamma \left| \frac{\partial u_{\epsilon,2}^k}{\partial t} \right|_1^2 + \frac{1}{2} \frac{d}{dt} |w_{\epsilon,2}^k|_1^2 \leq \left| \frac{\partial u_{\epsilon,2}^k}{\partial t} \right|_0^2 - D \left(\frac{\partial u_{\epsilon,1}^k}{\partial t}, \frac{\partial u_{\epsilon,2}^k}{\partial t} \right). \tag{2.3.3}$$

Adding (2.3.3) to (2.3.2) then using the Cauchy-Schwarz inequality and the Young's inequality (2.1.8)

$$\gamma \left(\left| \frac{\partial u_{\epsilon,1}^k}{\partial t} \right|_1^2 + \left| \frac{\partial u_{\epsilon,2}^k}{\partial t} \right|_1^2 \right) + \frac{1}{2} \frac{d}{dt} \left(|w_{\epsilon,1}^k|_1^2 + |w_{\epsilon,2}^k|_1^2 \right) \leq (1 + D) \left(\left| \frac{\partial u_{\epsilon,1}^k}{\partial t} \right|_0^2 + \left| \frac{\partial u_{\epsilon,2}^k}{\partial t} \right|_0^2 \right),$$

then noting (2.1.1a) and (2.1.5) to give

$$\begin{aligned}
&\gamma\left(\left|\frac{\partial u^k_{\epsilon,1}}{\partial t}\right|_1^2+\left|\frac{\partial u^k_{\epsilon,2}}{\partial t}\right|_1^2\right)+\frac{1}{2}\frac{d}{dt}\left(|w^k_{\epsilon,1}|_1^2)+|w^k_{\epsilon,2}|_1^2\right)\\
&\quad\le\frac{\gamma}{2}\left(\left|\frac{\partial u^k_{\epsilon,1}}{\partial t}\right|_1^2+\left|\frac{\partial u^k_{\epsilon,2}}{\partial t}\right|_1^2\right)+\frac{(1+D)^2}{2\gamma}\left(\left\|\frac{\partial u^k_{\epsilon,1}}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial u^k_{\epsilon,2}}{\partial t}\right\|_{-1}^2\right), \qquad (2.3.4)
\end{aligned}$$

so that, by (2.2.30), we have

$$\begin{aligned}
\frac{d}{dt}\left(|w^k_{\epsilon,1}|_1^2+|w^k_{\epsilon,2}|_1^2\right)+\gamma\left(\left|\frac{\partial u^k_{\epsilon,1}}{\partial t}\right|_1^2+\left|\frac{\partial u^k_{\epsilon,2}}{\partial t}\right|_1^2\right)&\le\frac{(1+D)^2}{\gamma}\left(\left\|\frac{\partial u^k_{\epsilon,1}}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial u^k_{\epsilon,2}}{\partial t}\right\|_{-1}^2\right)\\
&\le\frac{C}{\gamma}\left(|w^k_{\epsilon,1}|_1^2+|w^k_{\epsilon,2}|_1^2\right). \qquad (2.3.5)
\end{aligned}$$

Integrating over $(0,T)$ and using the Grönwall 's inequality yields

$$|w^k_{\epsilon,1}(t)|_1^2+|w^k_{\epsilon,2}(t)|_1^2\le\exp\left(\frac{Ct}{\gamma}\right)\left(|w^k_{\epsilon,1}(0)|_1^2+|w^k_{\epsilon,2}(0)|_1^2\right). \qquad (2.3.6)$$

We integrate the first term of (2.2.19b) by parts, use the boundary conditions, set $\eta=1$ and note (2.1.14b) which give, for $i,j=1,2,i\neq j$,

$$w^k_{\epsilon,i}(0)=-\gamma\Delta u^k_{\epsilon,i}(0)+P^k\psi'_\epsilon(u^k_{\epsilon,i}(0))+D(u^k_{\epsilon,j}(0)+1),$$

so that

$$|w^k_{\epsilon,i}(0)|_1\le|\gamma\Delta u^k_{\epsilon,i}(0)|_1+|\psi'_\epsilon(u^k_{\epsilon,i}(0))|_1+|D(u^k_{\epsilon,j}(0)+1)|_1. \qquad (2.3.7)$$

Noting (2.1.13), (2.1.14a) and setting $v=\Delta u^0_i$, then integrating the right-hand side by parts gives

$$\begin{aligned}
(P^k\Delta u^0_i,\eta^k)&=(\Delta u^0_i,\eta^k)\\
&=-(\nabla u^0_i,\nabla\eta^k)\\
&=-(\nabla P^k u^0_i,\nabla\eta^k)\\
&=(\Delta P^k u^0_i,\eta^k)
\end{aligned}$$

or

$$(P^k\Delta u^0_i-\Delta P^k u^0_i,\eta^k)=0\qquad\forall\eta^k\in V^k.$$

Choosing $\eta^k = P^k\Delta u_i^0 - \Delta P^k u_i^0 \in V^k$ to obtain

$$\|P^k\Delta u_i^0 - \Delta P^k u_i^0\|_0^2 = 0,$$

we thus have $P^k\Delta u_i^0 \equiv \Delta P^k u_i^0$ and noting (2.1.14b) yields

$$|\Delta u_{\epsilon,i}^k(0)|_1 = |\Delta P^k u_i^0|_1 = |P^k\Delta u_i^0|_1 \leq |\Delta u_i^0|_1. \tag{2.3.8}$$

Noting again (2.1.14b) we also have that

$$|u_{\epsilon,j}^k(0)+1|_1 = |u_{\epsilon,j}^k(0)|_1 = |P^k u_j^0|_1 \leq |u_j^0|_1. \tag{2.3.9}$$

Next we consider that

$$\begin{aligned} |\psi_\epsilon'(u_{\epsilon,i}^k(0))|_1 &= |\nabla\psi_\epsilon'(u_{\epsilon,i}^k(0))|_0 \\ &= |\psi_\epsilon''(u_{\epsilon,i}^k(0))\nabla u_{\epsilon,i}^k(0)|_0 \\ &\leq \|\psi_\epsilon''(u_{\epsilon,i}^k(0)\|_\infty |u_{\epsilon,i}^k(0)|_1 \\ &\leq \|\psi_\epsilon''(u_{\epsilon,i}^k(0))\|_\infty |u_i^0|_1. \end{aligned}$$

Also noting that

$$P^k u_i^0 \to u_i^0 \quad \text{in } L^2(\Omega) \qquad \text{or} \qquad P^k u_i^0 \to u_i^0 \quad a.e. \text{ in } \Omega,$$

and $|u_i^0| \leq 1-\delta$ *a.e* then, for k sufficiently large,

$$|P^k u_i^0 - u_i^0| \leq \frac{\delta}{2} \quad a.e.\,. \tag{2.3.10}$$

Hence we have that, for k sufficiently large,

$$|P^k u_i^0| \leq |P^k u_i^0 - u_i^0| + |u_i^0| \leq 1 - \frac{\delta}{2} \quad a.e.\,.$$

Noting (2.2.4) and (2.1.14b) to give

$$|\nabla\psi_\epsilon'(u_{\epsilon,i}^k(0)|_0 = |-\nabla u_{\epsilon,i}^k(0)|_0 = |-\nabla P^k u_i^0|_0 \leq |u_i^0|_1. \tag{2.3.11}$$

Combining (2.3.8), (2.3.9), and (2.3.11), then (2.3.7) yields that

$$|w_{\epsilon,i}^k(0)|_1 \leq C\Big(|\Delta u_i^0|_1 + |u_i^0|_1 + D|u_j^0|_1\Big).$$

Therefore we obtain

$$|w_{\epsilon,1}^k(0)|_1 + |w_{\epsilon,2}^k(0)|_1 \leq C\Big(\|u_1^0\|_3 + \|u_2^0\|_3\Big). \tag{2.3.12}$$

Together with the convergence properties of $u_{\epsilon,i}^k$ and $w_{\epsilon,i}^k$, (2.2.47) and (2.2.48a-d), the third bound in (2.3.1a) follows from (2.3.6) and (2.3.12). The first bound then follows from (2.3.5) and (2.3.6) such that

$$\int_0^T \gamma \exp\left(-\frac{Cs}{\gamma}\right)\left(\left|\frac{\partial u_{\epsilon,1}}{\partial t}\right|_1^2 + \left|\frac{\partial u_{\epsilon,2}}{\partial t}\right|_1^2\right) ds \leq \Big(|w_{\epsilon,1}(0)|_1^2 + |w_{\epsilon,2}(0)|_1^2\Big) \leq C.$$

And the second follows from the third and (2.3.5) such that

$$\frac{\gamma}{2}\left(\left|\frac{\partial u_{\epsilon,1}}{\partial t}\right|_1^2 + \left|\frac{\partial u_{\epsilon,2}}{\partial t}\right|_1^2\right) \leq C.$$

The first bound in (2.3.1b) follows from standard elliptic regularity such that

$$\|u_{\epsilon,i}\|_2 \leq C|\Delta u_{\epsilon,i}|_0 + \|u_{\epsilon,i}\|_1. \tag{2.3.13}$$

The second bound in (2.3.1b) follows from the first, integrating the first term of (2.2.17a) by parts, noting (2.2.2c) and standard elliptic regularity such that

$$\begin{aligned} \|w_{\epsilon,i}\|_2 &\leq C|\Delta w_{\epsilon,i}|_0 + \|w_{\epsilon,i}\|_1 \\ &= C\left|\frac{\partial u_{\epsilon,i}}{\partial t}\right|_0 + \|w_{\epsilon,i}\|_1. \end{aligned} \tag{2.3.14}$$

□

Corollary 2.3.2 For $i = 1, 2$, let $u_i^0 \in H^3(\Omega)$, $\frac{\partial u_i^0}{\partial \eta} = 0$ be such that $\|u_i^0\|_0 \leq 1 - \delta$. Let $d \leq 3$ with either Ω being a convex polyhedron or $\partial\Omega \in C^{1,1}$. Then the solution $\{u_1, u_2, w_1, w_2\}$ of Problem **(P)** is such that the following stability bounds hold :

$$\left\|\frac{\partial u_i}{\partial t}\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\frac{\partial u_i}{\partial t}\right\|_{L^\infty(0,T;(H^1(\Omega))')} + \|w_i\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \tag{2.3.15a}$$

$$\|u_i\|_{L^\infty(0,T;H^2(\Omega))} + \|w_i\|_{L^2(0,T;H^2(\Omega))} \leq C. \tag{2.3.15b}$$

Proof: From (2.3.1a,b), we pass to the limit to obtain (2.3.15a,b). □

Chapter 3

A Semi-discrete Approximation

The notation which is used in this chapter and Chapter 4 are introduced in section 3.1. The semi-discrete approximation to Problem (**P**) and Problem (**Q**) is formulated and also the existence and uniqueness are proven in Section 3.2. Error bounds of approximation solutions are presented in Section 3.3.

3.1 Notation and Results

We introduce the following notations relating to the finite element approximation :

1. $\mathcal{T}^h$ is a regular family of triangulations for Ω, see Ciarlet [21], consisting of closed simplices, τ, with maximum diameter not exceeding h, so that $\bar{\Omega} \equiv \cup_{\tau\in\mathcal{T}^h}\bar{\tau}$. If $\partial\Omega$ is curved then the boundary elements have at most one curved edge. Associated with $\mathcal{T}^h$ is the finite element space $S^h \subset H^1(\Omega)$

$$S^h \;=\; \{\chi \in C^0(\bar{\Omega}) : \chi|_\tau \text{ is linear for } \tau \in \mathcal{T}^h\}. \qquad (3.1.1)$$

2. $\pi^h : C(\bar{\Omega}) \mapsto S^h$ is the interpolation operator such that $\pi^h\chi(x_i) = \chi(x_i)$, for $i = 1, ..., N$ and define a discrete inner product on $C(\bar{\Omega})$ as follows

$$(\chi_1, \chi_2)^h \;=\; \int_\Omega \pi^h(\chi_1(x)\chi_2(x))dx \;\equiv\; \sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}\chi_1(x_i)\chi_2(x_j), \qquad (3.1.2)$$

where $M_{ij} = (\varphi_i, \varphi_j)^h = \int_\Omega \pi^h(\varphi_i(x)\varphi_j(x))dx = \int_\Omega \delta_{ij}\varphi_j(x)dx$ is the element of the $n\times n$ symmetric positive definite matrix and φ_i is the continuous piecewise linear function that takes the value 1 at node point x_i and the value 0 at other node points.

We introduce the discrete Green's operator $\widehat{\mathcal{G}}_N^h : \mathcal{F}^h \to V^h$ defined by

$$(\nabla \widehat{\mathcal{G}}_N^h v, \nabla \chi) = (v, \chi)^h \qquad \forall \chi \in S^h, \tag{3.1.3a}$$

$$(\widehat{\mathcal{G}}_N^h v, 1)^h = 0 \tag{3.1.3b}$$

where $V^h := \{v^h \in S^h : (v^h, 1) = 0\}$ and $\mathcal{F}^h := \{v \in C(\bar{\Omega}) : (v, 1)^h = 0\}$. We define the norm on $\mathcal{F}^h$ by

$$\|v\|_{-h}^2 \equiv |\widehat{\mathcal{G}}_N^h v|_1^2 = (\nabla \widehat{\mathcal{G}}_N^h v, \nabla \widehat{\mathcal{G}}_N^h v) = (v, \widehat{\mathcal{G}}_N^h v)^h \qquad \forall v \in \mathcal{F}^h. \tag{3.1.4}$$

Then on noting the Cauchy's inequality (2.1.9) yields, for all $\alpha > 0$,

$$\begin{aligned} (v, \chi)^h \equiv (\nabla \widehat{\mathcal{G}}_N^h v, \nabla \chi) &\leq |\widehat{\mathcal{G}}_N^h v|_1 |\chi|_1 \\ &= \|v\|_{-h} |\chi|_1 \\ &\leq \frac{1}{2\alpha}\|v\|_{-h} + \frac{\alpha}{2}|\chi|_1^2 \qquad \forall v \in \mathcal{F}^h, \chi \in S^h. \end{aligned} \tag{3.1.5}$$

We introduce also some well-known results concerning S^h:

for $\eta, \chi \in S^h$, $r = 0$ or 1, see also Cialvaldini [20],

$$|(\eta, \chi) - (\eta, \chi)^h| \leq Ch^{1+r} |\eta|_r |\chi|_1; \quad r = 0 \text{ or } 1, \tag{3.1.6a}$$

$$C_1 |\chi|_0^2 \leq |\chi|_h^2 \leq C_2 |\chi|_0^2 \tag{3.1.6b}$$

for $r = 0$ or 1 and for $p_1 \leq p_2 \leq \infty$, where $p_1 \geq 1$ if $d \leq 2$ or $p_1 > \frac{3}{2}$ if $d = 3$,

$$|(I - \pi^h)\eta|_{r,p_2} \leq Ch^{2-r-d(\frac{1}{p_1}-\frac{1}{p_2})} |\eta|_{2,p_1} \qquad \forall \eta \in W^{2,p_1}(\Omega), \tag{3.1.6c}$$

and the error estimate satisfied $\widehat{\mathcal{G}}_N^h$ and $\mathcal{G}_N$, see Barrett and Blowey [7],

$$|(\mathcal{G}_N - \widehat{\mathcal{G}}_N^h) v^h|_1 \leq Ch\,|v^h|_0 \qquad \forall v^h \in V^h \tag{3.1.6d}$$

where C, C_1 and C_2 are constants independent of h and W^{2,p_1} be the Sobolev space defined by $W^{k,p}(\Omega) = \{f \in L^p(\Omega) : \forall |\alpha| \leq k, \frac{\partial^\alpha f}{\partial x^\alpha} \in L^p(\Omega)\}$ for $d \leq 1$, Ω an open subset of $\mathbb{R}^d$, $p \in [1, \infty]$ and $k \in \mathbb{N}$. From (3.1.6b), we note that

$$|v^h|_h^2 \leq C\|v^h\|_1^2 \qquad \forall v^h \in V^h. \tag{3.1.7}$$

Noting (3.1.6a,b) and the continuous Poincaré inequality, for h sufficiently small we obtain the discrete Poincaré inequality

$$((\chi,\chi)^h)^{\frac{1}{2}} \equiv |\chi|_h \leq C_p\Big(|(\chi,1)^h| + |\chi|_1\Big), \tag{3.1.8}$$

where C_p is a constant independent of h. The same notation has been used for both the discrete and the continuous Poincaré constants C_p. Noting (3.1.8) and (3.1.3b) we have the analogue of (2.1.6) as

$$|\widehat{\mathcal{G}}^h_N v^h|_1^2 = (v^h, \widehat{\mathcal{G}}^h_N v^h)^h \leq |v^h|_h |\widehat{\mathcal{G}}^h_N v^h|_h \leq C_p |v^h|_h |\widehat{\mathcal{G}}^h_N v^h|_1. \tag{3.1.9}$$

Thus we derive the following useful inequality, the analogue of (2.1.7), as

$$\|v^h\|_{-h} \equiv |\widehat{\mathcal{G}}^h_N v^h|_1 \leq C_p |v^h|_h. \tag{3.1.10}$$

We note also the following useful inequalities, see Barrett and Blowey [7]

$$C_1 h^2 |v^h|_1 \leq C_2 h |v^h|_h \leq C_3 \|v^h\|_{-h} \leq C_4 \|v^h\|_{-1} \leq C_5 \|v^h\|_{-h} \quad \forall v^h \in V^h. \tag{3.1.11}$$

The first inequality on the left is the inverse inequality on noting (3.1.6b) and holds for all $v^h \in S^h$. The second inequality follows from the first and (3.1.5). The third and fourth inequalities follow from noting (3.1.9), (2.1.4), (2.1.1a), (3.1.3a), and (3.1.6a).

Our choice of initial data for the finite element approximation $u_i^{h,0}$ is open. However, for ease of exposition we take

$$u_i^{h,0} = P^h u_i^0$$

where P^h is the discrete L^2 projection onto S^h, $P^h : L^2(\Omega) \to S^h$, as

$$(P^h \eta, \chi)^h = (\eta, \chi) \qquad \forall \chi \in S^h, \eta \in L^2(\Omega).$$

Actually it is sufficient that $u_i^{h,0}$ satisfies

$$u_i^{h,0} \in K^h_{m_i}, \tag{3.1.12}$$

$$\|u_i^{h,0} - u_i^0\|_{-1} \leq Ch, \tag{3.1.13}$$

where $K^h_{m_i} := \{\eta \in K^h : (\eta, 1)^h = (u_i^0, 1)^h = m\}$ and $K^h := \{\eta \in S^h : -1 \le \eta \le 1\}$. Clearly $P^h u_i^0 \in K^h_{m_i}$ since $P^h u_i^0 \in S^h$,

$$(P^h u_i^0, 1)^h = (u_i^0, 1) = m_i$$

and

$$\begin{aligned} M_{jj}|P^h u_i^0(x_j)| &= |(P^h u_i^0, \varphi_j)^h| \\ &= |(u_i^0, \varphi_j)| \le (1, \varphi_j) = M_{jj}. \end{aligned}$$

Finally (3.1.13) follows from page 87 in Blowey [10]. In Chapter 4, Section 4.3, we choose an alternative for $u_i^{h,0}$ which has additional key features necessary for the error analysis.

3.2 Existence and Uniqueness

We introduce the following semi-discrete finite element approximation of Problem **(P)** (2.2.1a-c) and Problem **(Q)** (2.2.2a,b):

Problem **($\mathbf{P}^h$)**: For $\gamma, D > 0$ find $\{u_1^h, u_2^h, w_1^h, w_2^h\} \in K^h \times K^h \times S^h \times S^h$ such that for $a.e.\, t \in (0,T)$, $i, j = 1, 2, i \ne j$,

$$\left(\frac{\partial u_i^h}{\partial t}, \eta\right)^h + (\nabla w_i^h, \nabla \eta) = 0, \qquad \eta \in S^h \tag{3.2.1a}$$

$$\gamma(\nabla u_i^h, \nabla\eta - \nabla u_i^h) - (u_i^h, \eta - u_i^h)^h + D(u_j^h + 1, \eta - u_i^h)^h \ge (w_i^h, \eta - u_i^h)^h \; \forall \eta \in S^h, \tag{3.2.1b}$$

$$u_i^h(\cdot, 0) = u_i^{h,0}(\cdot). \tag{3.2.1c}$$

Problem **($\mathbf{Q}^h$)**: For $\gamma, D > 0$ find $\{u_1^h, u_2^h\} \in K^h_{m_1} \times K^h_{m_2}$, such that for $a.e.\, t \in (0,T)$, $i = 1, 2$, $i \ne j$,

$$\gamma(\nabla u_i^h, \nabla\eta - \nabla u_i^h) + \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, \eta - u_i^h\right)^h - (u_i^h, \eta - u_i^h)^h + D(u_j^h + 1, \eta - u_i^h)^h \ge 0,$$
$$\forall \eta \in K^h_{m_i} \tag{3.2.2a}$$

$$u_i^h(\cdot,0) \;=\; u_i^{h,0}(\cdot) \in K_{m_i}^h. \tag{3.2.2b}$$

Noting (2.2.4) - (2.2.12), we introduce also the following penalized Problem $(\mathbf{P}_\epsilon^h)$ as

Problem $(\mathbf{P}_\epsilon^h)$: For $\epsilon, \gamma, D > 0$ find $\{u_{\epsilon,1}^h, u_{\epsilon,2}^h, w_{\epsilon,1}^h, w_{\epsilon,2}^h\} \in S^h \times S^h \times S^h \times S^h$, such that for $a.e.\, t \in (0,T)$, $i,j = 1,2$, $i \neq j$,

$$\left(\frac{\partial u_{\epsilon,i}^h}{\partial t}, \eta\right)^h + (\nabla w_{\epsilon,i}^h, \nabla \eta) \;=\; 0, \qquad \forall \eta \in S^h \tag{3.2.3a}$$

$$\gamma(\nabla u_{\epsilon,i}^h, \nabla\eta) + (\psi_\epsilon'(u_{\epsilon,i}^h), \eta)^h + D(u_{\epsilon,j}^h + 1, \eta)^h \;=\; (w_{\epsilon,i}^h, \eta)^h, \quad \forall \eta \in S^h \tag{3.2.3b}$$

$$u_{\epsilon,i}^h(\cdot,0) \;=\; u_i^{h,0}(\cdot). \tag{3.2.3c}$$

Problem $(\mathbf{Q}_\epsilon^h)$: For $\epsilon, \gamma, D > 0$ find $\{u_{\epsilon,1}^h, u_{\epsilon,2}^h\} \in S^h \times S^h$, such that for $a.e.\, t \in (0,T)$,

$i,j = 1,2$, $i \neq j$,

$$\gamma(\nabla u_{\epsilon,i}^h, \nabla\eta) + \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_{\epsilon,i}^h}{\partial t}, \eta\right)^h + (\psi_\epsilon'(u_{\epsilon,i}^h), \eta)^h + D(u_{\epsilon,j}^h + 1, \eta)^h \;=\; 0, \quad \forall \eta \in S_m^h \tag{3.2.4a}$$

$$u_{\epsilon,i}^h(\cdot,0) \;=\; u_i^{h,0}(\cdot), \tag{3.2.4b}$$

with

$$w_{\epsilon,i}^h \;=\; -\widehat{\mathcal{G}}_N^h \frac{\partial u_{\epsilon,i}^h}{\partial t} + \lambda^h \in S^h, \; \lambda^h \;=\; \fint \left(\psi_\epsilon'(u_{\epsilon,i}^h) + u_{\epsilon,j}^h + 1\right). \tag{3.2.5}$$

Taking $u_i^0 = P^h u_i^0$ we have the following

Theorem 3.2.1 There exists a unique solution $\{u_{\epsilon,1}^h, u_{\epsilon,2}^h, w_{\epsilon,1}^h, w_{\epsilon,2}^h\}$ to Problem $(\mathbf{P}_\epsilon^h)$ such that the following stability bounds hold

$$\|u_{\epsilon,i}^h\|_{H^1(0,T;(H(\Omega))')} \;\leq\; C, \tag{3.2.6a}$$

$$\|u_{\epsilon,i}^h\|_{L^\infty(0,T;H^1(\Omega))} \;\leq\; C, \tag{3.2.6b}$$

$$\|w_{\epsilon,i}^h\|_{L^2(0,T;H^1(\Omega))} \;\leq\; C\left(1 + T^{\frac{1}{2}}\right). \tag{3.2.6c}$$

Proof: We introduce the approximation to Problem $(\mathbf{P}_\epsilon^h)$ as

Find $\{u_{\epsilon,1}^h, u_{\epsilon,2}^h, w_{\epsilon,1}^h, w_{\epsilon,2}^h,\} \in S^h \times S^h \times S^h \times S^h$

$$u_{\epsilon,i}^h(x,t) \;=\; \sum_{m=1}^N c_{i,m}(t)\varphi_m(x) \;\text{ and }\; w_{\epsilon,i}^h(x,t) \;=\; \sum_{n=1}^N d_{i,m}(t)\varphi_m(x);$$

such that (3.2.3a-c) is satisfied.

Choosing $\eta^h = \varphi_n, n = 1, ..., N$ in (3.2.3a,b)

$$\sum_{m=1}^{N} \frac{d\, c_{im}}{dt}(\varphi_m, \varphi_n)^h + \sum_{m=1}^{N} d_{im}(\nabla\varphi_m, \nabla\varphi_n) = 0, \tag{3.2.7a}$$

$$\gamma \sum_{m=1}^{N} c_{im}(\nabla\varphi_m, \nabla\varphi_n) + \frac{1}{\epsilon}(\beta_\epsilon(u^h_{\epsilon,i}), \varphi_n)^h - \sum_{m=1}^{N} c_{im}(\varphi_m, \varphi_n)^h$$
$$+ D\sum_{m=1}^{N} c_{jm}(\varphi_m, \varphi_n)^h + \sum_{m=1}^{N} 1(\varphi_m, \varphi_n)^h = \sum_{m=1}^{N} d_{im}(\varphi_m, \varphi_n)^h, \tag{3.2.7b}$$

$$\sum_{m=1}^{N} c_{im}(0)(\varphi_m, \varphi_n)^h = (P^h u_i^0, \varphi_n)^h =: b_n, \tag{3.2.7c}$$

note that as $(\varphi_m, \varphi_n)^h = \delta_{mn}|\varphi_m|_h^2$ then $b_n = c_{i,n} = (u_i^0, \varphi_n)$.

We also define that

$$\begin{aligned}
\{\mathbf{M}\}_{mn} &= (\varphi_m, \varphi_n)^h, \\
\{\mathbf{A}\}_{mn} &= (\nabla\varphi_m, \nabla\varphi_n), \\
\{\mathbf{f}(c_i)\}_n &= \frac{1}{\epsilon}(\beta_\epsilon(u^h_{\epsilon,i}), \varphi_n)^h.
\end{aligned}$$

Therefore (3.2.7a-c) can be written as

$$\begin{aligned}
\mathbf{M}\frac{d\mathbf{c}_i}{dt} + \mathbf{A}\mathbf{d}_i &= 0, \\
\gamma\mathbf{A}\mathbf{c}_i + \mathbf{f}(\mathbf{c}_i) - \mathbf{M}\mathbf{c}_i + D\mathbf{M}\mathbf{c}_i + D\mathbf{M}\underline{1} &= \mathbf{M}\mathbf{d}_i, \\
\mathbf{M}\mathbf{c}_i(0) &= \mathbf{b},
\end{aligned}$$

which is equivalent to the following system of ordinary differential equations

$$\frac{d\mathbf{c}_i}{dt} = -\gamma\mathbf{M}^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{A}\mathbf{c}_i - \mathbf{M}^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{f}(\mathbf{c}_i) + \mathbf{M}^{-1}\mathbf{A}\mathbf{c}_i - D\mathbf{M}^{-1}\mathbf{A}\mathbf{c}_i + D\mathbf{M}^{-1}\mathbf{A}\underline{1},$$

$$\mathbf{M}\mathbf{c}_i(0) = \mathbf{b}.$$

Defining $\widehat{\mathbf{c}} = [\mathbf{c}_1, \mathbf{c}_2]^T$ and $\widehat{\mathbf{u}}^h = [\mathbf{u}_1^h, \mathbf{u}_2^h]^T$, we obtain systems of ordinary differential equations

$$\frac{d\widehat{\mathbf{c}}}{dt} = \mathcal{H}(\widehat{\mathbf{c}}),$$

$$\widehat{\mathbf{c}}(0) = \mathcal{M}\widehat{\mathbf{b}}.$$

Then applying the existence theorem of a system of ordinary differential equations, see appendix, there is existence of $u^h_{\epsilon,i}$ and $w^h_{\epsilon,i}$ as $\mathcal{H}$ is globally Lipschitz. We now calculate some a *priori* estimates.

We first consider the second bound (3.2.6b). For $i,j = 1,2, i \neq j$, setting $\eta = \frac{\partial u^h_{\epsilon,i}}{\partial t}$ in (3.2.4a) and noting $\left(\frac{\partial u^h_{\epsilon,i}}{\partial t}, 1\right)^h = 0$, we then have

$$\gamma\left(\nabla u^h_{\epsilon,1}, \nabla\frac{\partial u^h_{\epsilon,1}}{\partial t}\right)$$
$$+\left(\widehat{\mathcal{G}}^h_N\frac{\partial u^h_{\epsilon,1}}{\partial t}, \frac{\partial u^h_{\epsilon,1}}{\partial t}\right)^h + \left(\psi'_\epsilon(u^h_{\epsilon,1}), \frac{\partial u^h_{\epsilon,1}}{\partial t}\right)^h + D\left(u^h_{\epsilon,2}+1, \frac{\partial u^h_{\epsilon,1}}{\partial t}\right)^h = 0, \quad (3.2.8)$$

and

$$\gamma\left(\nabla u^h_{\epsilon,2}, \nabla\frac{\partial u^h_{\epsilon,2}}{\partial t}\right)$$
$$+\left(\widehat{\mathcal{G}}^h_N\frac{\partial u^h_{\epsilon,2}}{\partial t}, \frac{\partial u^h_{\epsilon,2}}{\partial t}\right)^h + \left(\psi'_\epsilon(u^h_{\epsilon,2}), \frac{\partial u^h_{\epsilon,2}}{\partial t}\right)^h + D\left(u^h_{\epsilon,1}+1, \frac{\partial u^h_{\epsilon,2}}{\partial t}\right)^h = 0. \quad (3.2.9)$$

Adding (3.2.8) to (3.2.9), rearranging the terms and integrating over $(0,t)$, we have, for all $t \in (0,T)$

$$\begin{aligned}
0 &= \gamma\int_0^t \left(\nabla u^h_{\epsilon,1}, \nabla\frac{\partial u^h_{\epsilon,1}}{\partial s}\right) ds + \gamma\int_0^t \left(\nabla u^h_{\epsilon,2}, \nabla\frac{\partial u^h_{\epsilon,2}}{\partial s}\right) ds \\
&+ \int_0^t \left(\widehat{\mathcal{G}}^h_N\frac{\partial u^h_{\epsilon,1}}{\partial t}, \frac{\partial u^h_{\epsilon,1}}{\partial s}\right)^h ds + \int_0^t \left(\widehat{\mathcal{G}}^h_N\frac{\partial u^h_{\epsilon,2}}{\partial t}, \frac{\partial u^h_{\epsilon,2}}{\partial s}\right)^h ds \\
&+ \int_0^t \left(\psi'_\epsilon(u^h_{\epsilon,1}), \frac{\partial u^h_{\epsilon,1}}{\partial s}\right)^h ds + \int_0^t \left(\psi'_\epsilon(u^h_{\epsilon,2}), \frac{\partial u^h_{\epsilon,2}}{\partial s}\right)^h ds \\
&+ \int_0^t D\left(u^h_{\epsilon,2}+1, \frac{\partial u^h_{\epsilon,1}}{\partial s}\right)^h ds + \int_0^t D\left(u^h_{\epsilon,1}+1, \frac{\partial u^h_{\epsilon,2}}{\partial s}\right)^h ds. \qquad (3.2.10)
\end{aligned}$$

For $i = 1,2$, we consider each term on the right-hand side of (3.2.10). The first and second terms, noting (3.2.1c), becomes

$$\begin{aligned}
\int_0^t \left(\nabla u^h_{\epsilon,i}, \nabla\frac{\partial u^h_{\epsilon,i}}{\partial s}\right) ds &= \frac{1}{2}\int_0^t \frac{\partial}{\partial s}(\nabla u^h_{\epsilon,i}, \nabla u^h_{\epsilon,i})\, ds \\
&= \frac{1}{2}|u^h_{\epsilon,i}(t)|_1^2 - \frac{1}{2}|u^h_{\epsilon,i}(0)|_1^2 \\
&= \frac{1}{2}|u^h_{\epsilon,i}(t)|_1^2 - \frac{1}{2}|P^h u^0_i|_1^2. \qquad (3.2.11)
\end{aligned}$$

Noting (3.1.4), we obtain the third and fourth terms as

$$\int_0^t \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_{\epsilon,i}^h}{\partial s}, \frac{\partial u_{\epsilon,i}^h}{\partial s}\right)^h ds = \int_0^t \left|\widehat{\mathcal{G}}_N^h \frac{\partial u_{\epsilon,i}^h}{\partial s}\right|_1^2 ds = \int_0^t \left\|\frac{\partial u_{\epsilon,i}^h}{\partial s}\right\|_{-h}^2 ds. \tag{3.2.12}$$

Noting (3.1.2) the fifth and sixth terms can be expressed as

$$\begin{aligned}
\int_0^t (\psi_\epsilon'(u_{\epsilon,i}^h), \frac{\partial u_{\epsilon,i}^h}{\partial s})^h ds &= \int_\Omega \pi^h \left(\int_0^t \psi_\epsilon'(u_{\epsilon,i}^h)\frac{\partial u_{\epsilon,i}^h}{\partial s} ds\right) dx \\
&= \int_\Omega \pi^h \left(\int_0^t \frac{\partial}{\partial s}\psi_\epsilon(u_{\epsilon,i}^h(s))\, ds\right) dx \\
&= \int_\Omega \pi^h \Big(\psi_\epsilon(u_{\epsilon,i}^h(t)) - \psi_\epsilon(u_{\epsilon,i}^h(0))\Big)\, dx \\
&= (\psi_\epsilon(u_{\epsilon,i}^h(t)), 1)^h - (\psi_\epsilon(u_{\epsilon,i}^h(0)), 1)^h,
\end{aligned} \tag{3.2.13}$$

and the last two terms can be written as

$$\begin{aligned}
&\int_0^t \left(u_{\epsilon,2}^h + 1, \frac{\partial u_{\epsilon,1}^h}{\partial s}\right)^h ds + \int_0^t \left(u_{\epsilon,1}^h + 1, \frac{\partial u_{\epsilon,2}^h}{\partial s}\right)^h ds \\
&= \int_0^t \left(\int_\Omega \pi^h \left((u_{\epsilon,2}^h + 1)\frac{\partial u_{\epsilon,1}^h}{\partial s} + (u_{\epsilon,1}^h + 1)\frac{\partial u_{\epsilon,2}^h}{\partial s}\right) dx\right) ds \\
&= \int_\Omega \pi^h \Big(\int_0^t \frac{\partial (u_{\epsilon,1}^h + 1)(u_{\epsilon,2}^h + 1)}{\partial s} ds\Big)\, dx \\
&= \int_\Omega \Big[(u_{\epsilon,1}^h(t) + 1)(u_{\epsilon,2}^h(t) + 1) - (u_{\epsilon,1}^h(0) + 1)(u_{\epsilon,2}^h(0) + 1)\Big]\, dx \\
&= (u_{\epsilon,1}^h(t) + 1, u_{\epsilon,2}^h(t) + 1)^h - (u_{\epsilon,1}^h(0) + 1, u_{\epsilon,2}^h(0) + 1)^h.
\end{aligned} \tag{3.2.14}$$

Substitute (3.2.11) - (3.2.14) into (3.2.10) then rearrange the terms to obtain

$$\begin{aligned}
&\frac{\gamma}{2}\Big(|u_{\epsilon,1}^h(t)|_1^2 + |u_{\epsilon,2}^h(t)|_1^2\Big) + (\psi_\epsilon(u_{\epsilon,1}^h(t)) + \psi_\epsilon(u_{\epsilon,2}^h(t)), 1)^h + D(u_{\epsilon,1}^h(t) + 1, u_{\epsilon,2}^h(t) + 1)^h \\
&+ \int_0^t \left(\left\|\frac{\partial u_{\epsilon,1}^h(t)}{\partial s}\right\|_{-h}^2 + \left\|\frac{\partial u_{\epsilon,2}^h(t)}{\partial s}\right\|_{-h}^2\right) ds \\
&= \frac{\gamma}{2}\Big(|P^h u_1^0|_1^2 + |P^h u_2^0|_1^2\Big) + (\psi_\epsilon(u_{\epsilon,1}^h(0)) + \psi_\epsilon(u_{\epsilon,2}^h(0)), 1)^h + D(u_{\epsilon,1}^h(0) + 1, u_{\epsilon,2}^h(0) + 1)^h.
\end{aligned} \tag{3.2.15}$$

Noting (2.2.3) gives

$$\begin{aligned}
0 \le (\psi(u_{\epsilon,i}^h(0)), 1)^h &= \frac{1}{2}(1 - (u_{\epsilon,i}^h(0))^2, 1)^h \\
&\le \frac{1}{2}|\Omega|.
\end{aligned} \tag{3.2.16}$$

Noting (3.1.7), (3.2.1c), inequality $2ab \le a^2+b^2$ and $(a+b)^2 \le 2(a^2+b^2)$, we obtain

$$\begin{aligned} 0 &\le (u^h_{\epsilon,1}(0)+1, u^h_{\epsilon,2}(0)+1)^h \\ &\le |u^h_{\epsilon,1}(0)+1|_h |u^h_{\epsilon,2}+1|_h \\ &\le C\|P^h u^0_1\|^2_1 + C\|P^h u^0_2\|^2_1 + 2|\Omega|. \end{aligned} \tag{3.2.17}$$

In the same way as (2.2.13) we have that

$$(\psi_\epsilon(u^h_{\epsilon,1}, 1)^h + (\psi_\epsilon(u^h_{\epsilon,2}, 1)^h + D((u^h_{\epsilon,1}+1)(u^h_{\epsilon,2}+1), 1)^h \ \ge\ -C. \tag{3.2.18}$$

Substituting (3.2.16) and (3.2.17) into (3.2.15), then noting (3.2.18) yields

$$\begin{aligned} &\frac{\gamma}{2}\Big(|u^h_{\epsilon,1}(t)|^2_1 + |u^h_{\epsilon,2}(t)|^2_1\Big) \ +\ \int_0^t \left(\left\|\frac{\partial u^h_{\epsilon,1}}{\partial s}\right\|^2_{-h} + \left\|\frac{\partial u^h_{\epsilon,2}}{\partial s}\right\|^2_{-h} \right) ds \\ &\quad +\ (\psi_\epsilon(u^h_{\epsilon,1}(t), 1)^h + (\psi_\epsilon(u^h_{\epsilon,2}(t), 1)^h \ +\ D((u^h_{\epsilon,1}(t)+1), (u^h_{\epsilon,2}(t)+1))^h \\ &\le \frac{\gamma}{2}|P^h u^0_1|^2_1 + \frac{\gamma}{2}|P^h u^0_2|^2_1 + C\|P^h u^0_1\|^2_1 + C\|P^h u^0_2\|^2_1 + 3|\Omega| \\ &\le C. \end{aligned} \tag{3.2.19}$$

Using the discrete Poincaré inequatlity (3.1.8) and (3.2.19) gives

$$|u^h_{\epsilon,i}(t)|_0 \ \le\ C_p\Big(|u^h_{\epsilon,i}(t)|_1 + |(u^h_{\epsilon,i}(t), 1)^h|\Big) \ \le\ C,$$

then it follows that

$$\|u^h_{\epsilon,i}(t)\|_{L^\infty(0,T;H^1(\Omega))} \ \le\ C. \tag{3.2.20}$$

Setting $t = T$ in (3.2.19) we obtain

$$\begin{aligned} &\frac{\gamma}{2}\Big(|u^h_{\epsilon,1}(T)|^2_1 + |u^h_{\epsilon,2}(T)|^2_1\Big) \ +\ \int_0^T \left(\left\|\frac{\partial u^h_{\epsilon,1}}{\partial s}\right\|^2_{-h} + \left\|\frac{\partial u^h_{\epsilon,2}}{\partial s}\right\|^2_{-h} \right) ds \\ &\quad +\ (\psi_\epsilon(u^h_{\epsilon,1}(T), 1)^h + (\psi_\epsilon(u^h_{\epsilon,2}(T)), 1)^h \ +\ D((u^h_{\epsilon,1}(T)+1), (u^h_{\epsilon,2}(T)+1))^h \\ &\le C. \end{aligned} \tag{3.2.21}$$

Therefore we have that

$$\int_0^t \left(\left\|\frac{\partial u^h_{\epsilon,1}}{\partial s}\right\|^2_{-h} + \left\|\frac{\partial u^h_{\epsilon,2}}{\partial s}\right\|^2_{-h} \right) ds \ \le\ C. \tag{3.2.22}$$

Noting (3.1.11), we hence obtain

$$\left\|\frac{\partial u^h_{\epsilon,i}}{\partial t}\right\|_{L^2(0,T;(H^1(\Omega))')} \leq C.$$

Now we consider

$$\begin{aligned}\|u^h_{\epsilon,i} - ⨍\, u^h_{\epsilon,i}\|^2_{-h} &= \left\|u^h_{\epsilon,i}(t) - u^h_{\epsilon,i}(0) + u^h_{\epsilon,i}(0) - \frac{1}{|\Omega|}(u^h_{\epsilon,i}(t),1)\right\|^2_{-h} \\ &= \left\|\int_0^t \frac{\partial}{\partial s}u^h_{\epsilon,i}(t)\,ds + u^h_{\epsilon,i}(0) - \frac{1}{|\Omega|}(u^h_{\epsilon,i}(t),1)\right\|^2_{-h}.\end{aligned}$$

Noting $2ab \leq a^2 + b^2$, (3.1.10), (3.1.6a) and (3.2.22) yields

$$\begin{aligned}\|u^h_{\epsilon,i} - ⨍\, u^h_{\epsilon,i}\|^2_{-h} &\leq 2\left\|\int_0^t \frac{\partial}{\partial s}u^h_{\epsilon,i}(s)\,ds\right\|^2_{-h} + 2\left\|u^h_{\epsilon,i}(0) - \frac{1}{|\Omega|}(u^h_{\epsilon,i}(t),1)\right\|^2_{-h} \\ &\leq 2\left\|\int_0^t \frac{\partial}{\partial s}u^h_{\epsilon,i}(s)\,ds\right\|^2_{-h} + C|u^h_{\epsilon,i}(0)|^2_0 + C|(u^h_{\epsilon,i}(t),1)|^2_0 \\ &\leq 2\left\|\int_0^t \frac{\partial}{\partial s}u^h_{\epsilon,i}(s)\,ds\right\|^2_{-h} + C \\ &\leq C. \end{aligned} \tag{3.2.23}$$

Integrating (3.2.23) over $(0,T)$, then

$$\|u^h_{\epsilon,i} - ⨍\, u^h_{\epsilon,i}\|_{L^2(0,T;(H^1(\Omega))')} \leq C(T) \leq C. \tag{3.2.24}$$

Hence it follows from (3.2.22) and (3.2.24) that

$$\|u^h_{\epsilon,i}\|_{H^1(0,T;(H^1(\Omega))')} \leq C.$$

We use the discrete Poincaré inequality (3.1.8) to obtain

$$|w^h_{\epsilon,i}(t)|_h \leq C_p\Big(|w^h_{\epsilon,i}(t)|_1 + |(w^h_{\epsilon,i}(t),1)|\Big),$$

and then note the Young's inequality (2.1.8) to have

$$|w^h_{\epsilon,i}(t)|^2_h \leq C\Big(|w^h_{\epsilon,i}(t)|^2_1 + |(w^h_{\epsilon,i}(t),1)|^2\Big). \tag{3.2.25}$$

It follows from the definition of the norm in H^1, and (3.1.6b) that

$$\|w^h_{\epsilon,i}(t)\|^2_1 = |w^h_{\epsilon,i}(t)|^2_0 + |w^h_{\epsilon,i}(t)|^2_1,$$

and using (3.2.25) we have

$$\|w_{\epsilon,i}^h(t)\|_1^2 \le C\Big(|w_{\epsilon,i}^h(t)|_1^2 + |(w_{\epsilon,i}^h(t),1)|^2\Big). \tag{3.2.26}$$

Noting (3.2.5) and (3.1.4) gives

$$|w_{\epsilon,i}^h|_1^2 = \left|-\widehat{\mathcal{G}}_N^h \frac{\partial u_{\epsilon,i}^h}{\partial t} + \lambda_i\right|_1^2 = \left|\widehat{\mathcal{G}}_N^h \frac{\partial u_{\epsilon,i}^h}{\partial t}\right|_1^2 = \left\|\frac{\partial u_{\epsilon,i}^h}{\partial t}\right\|_{-h}^2. \tag{3.2.27}$$

Set $\eta = 1$ in (3.2.3b) and note $\psi_\epsilon'(u_{\epsilon,i}^h) = \frac{1}{\epsilon}\beta_\epsilon(u_{\epsilon,i}^h) - u_{\epsilon,i}^h$ to obtain

$$(w_{\epsilon,i}^h, 1)^h = D(u_{\epsilon,j}^h + 1, 1)^h - (u_{\epsilon,i}^h, 1)^h + \frac{1}{\epsilon}(\beta_\epsilon(u_{\epsilon,i}^h), 1)^h,$$

which implies that

$$|(w_{\epsilon,i}^h, 1)^h| \le \left|\frac{1}{\epsilon}(\beta_\epsilon(u_{\epsilon,i}^h), 1)^h\right| + |(u_{\epsilon,i}^h, 1)^h| + |D(u_{\epsilon,j}^h + 1, 1)^h|.$$

Noting (2.2.6) we have that $\beta_\epsilon(r) \equiv 0$ for $[-1,1]$ and $|\beta_\epsilon(r)| \le r\beta_\epsilon(r)$ for $|r| \ge 1$, then

$$\frac{1}{\epsilon}|(\beta_\epsilon(u_{\epsilon,i}^h), 1)^h| \le \frac{1}{\epsilon}(\beta_\epsilon(u_{\epsilon,i}^h), u_{\epsilon,i}^h)^h.$$

Hence it follows that

$$|(w_{\epsilon,i}^h(t), 1)| \le \frac{1}{\epsilon}(\beta_\epsilon(u_{\epsilon,i}^h), u_{\epsilon,i}^h)^h + |(u_{\epsilon,i}^h, 1)^h| + |D(u_{\epsilon,j}^h + 1, 1)^h|. \tag{3.2.28}$$

Substitute $\eta = u_{\epsilon,i}^h$ in (3.2.3b) and noting (2.2.6) and that $\psi_\epsilon'(u_{\epsilon,i}^h) = \frac{1}{\epsilon}\beta_\epsilon(u_{\epsilon,i}^h) - u_{\epsilon,i}^h$, we have

$$\begin{aligned}
\frac{1}{\epsilon}(\beta_\epsilon(u_{\epsilon,i}^h), u_{\epsilon,i}^h)^h &= (u_{\epsilon,i}^h, u_{\epsilon,i}^h)^h - \gamma(\nabla u_{\epsilon,i}^h, \nabla u_{\epsilon,i}^h) - D(u_{\epsilon,j}^h + 1, u_{\epsilon,i}^h)^h + (w_{\epsilon,i}^h, u_{\epsilon,i}^h)^h \\
&= |u_{\epsilon,i}^h|_h^2 - \gamma|u_{\epsilon,i}^h|_1^2 - D(u_{\epsilon,j}^h + 1, u_{\epsilon,i}^h)^h + (w_{\epsilon,i}^h, u_{\epsilon,i}^h)^h,
\end{aligned}$$

thus (3.2.28) becomes

$$|(w_{\epsilon,i}^h, 1)^h| \le |u_{\epsilon,i}^h|_h^2 - \gamma|u_{\epsilon,i}^h|_1^2 - D(u_{\epsilon,j}^h+1, u_{\epsilon,i}^h)^h + (w_{\epsilon,i}^h, u_{\epsilon,i}^h)^h + |(u_{\epsilon,i}^h, 1)^h| + |D(u_{\epsilon,j}^h+1, 1)^h|. \tag{3.2.29}$$

Note (3.1.3a) and also $(u_{\epsilon,i}^h, 1)^h = m_i$ to have

$$\begin{aligned}
(w_{\epsilon,i}^h, u_{\epsilon,i}^h)^h &= \left(w_{\epsilon,i}^h, u_{\epsilon,i}^h - \frac{1}{|\Omega|}(u_{\epsilon,i}^h, 1)^h\right)^h + \frac{1}{|\Omega|}(u_{\epsilon,i}^h, 1)^h (w_{\epsilon,i}^h, 1)^h \\
&= \left(\nabla w_{\epsilon,i}^h, \nabla \widehat{\mathcal{G}}_N^h\left(u_{\epsilon,i}^h - \frac{1}{|\Omega|}(u_{\epsilon,i}^h, 1)^h\right)\right) + \frac{m_i}{|\Omega|}(w_{\epsilon,i}^h, 1)^h \\
&\le |w_{\epsilon,i}^h|_1 \left|\widehat{\mathcal{G}}_N^h\left(u_{\epsilon,i}^h - \frac{m_i}{|\Omega|}\right)\right|_1 + \frac{m_i}{|\Omega|}(w_{\epsilon,i}^h, 1)^h.
\end{aligned} \tag{3.2.30}$$

As $|(u^h_{\epsilon,i},1)^h| = |m_i| < |\Omega|$, rearranging (3.2.29), and using (3.2.30) and (3.1.10), we obtain

$$\begin{aligned}
&|(w^h_{\epsilon,i},1)^h| \\
&\quad\leq\ |u^h_{\epsilon,i}|^2_h - \gamma|u^h_{\epsilon,i}|^2_1 - D(u^h_{\epsilon,j}+1,u^h_{\epsilon,i})^h + |m_i| + D|m_j + |\Omega|| + |w^h_{\epsilon,i}|_1 \left\| u^h_{\epsilon,i} - \frac{m_i}{|\Omega} \right\|_{-h} \\
&\qquad + \frac{m_i}{|\Omega|}(w^h_{\epsilon,i},1)^h \\
&\quad\leq\ |u^h_{\epsilon,i}|^2_h - D(u^h_{\epsilon,j}+1,u^h_{\epsilon,i})^h + |m_i| + D|m_i + |\Omega|| + C_p|w^h_{\epsilon,i}|_1 \left| u^h_{\epsilon,i} - \frac{m_i}{|\Omega|} \right|_h \\
&\qquad + \frac{|m_i|}{|\Omega|}|(w^h_{\epsilon,i},1)^h| \\
&\quad\leq\ \frac{|m_i| + |u^h_{\epsilon,i}|^2_h + C_p|w^h_{\epsilon,i}|_1 \left| u^h_{\epsilon,i} - \frac{m_i}{|\Omega|}\right|_h - D(u^h_{\epsilon,j}+1,u^h_{\epsilon,i})^h + D|m_i + |\Omega||}{1 - \frac{|m_i|}{|\Omega|}}.
\end{aligned} \tag{3.2.31}$$

Noting the Young's inequality (2.1.8) and (3.2.20), we have

$$\begin{aligned}
\left|D(u^h_{\epsilon,j}+1,u^h_{\epsilon,i})^h\right| &= \left|D\Big((u^h_{\epsilon,j},u^h_{\epsilon,i})^h + m_i\Big)\right| \\
&\leq D\left(\frac{1}{2}|u^h_{\epsilon,j}|^2_h + \frac{1}{2}|u^h_{\epsilon,i}|^2_h + m_i\right)
\end{aligned} \tag{3.2.32}$$

Noting (3.2.32), (3.2.31) becomes

$$|(w^h_{\epsilon,i},1)^h| \ \leq\ C\Big(|w^h_{\epsilon,i}|_1 + 1\Big).$$

Noting inequality $(a+b)^2 \leq 2(a^2+b^2)$, then

$$|(w^h_{\epsilon,i},1)^h|^2 \ \leq\ C\Big(|w^h_{\epsilon,i}|^2_1 + 1\Big). \tag{3.2.33}$$

Substituting (3.2.33) into (3.2.26) yields

$$\|w^h_{\epsilon,i}(t)\|^2_1 \ \leq\ C\Big(|w^h_{\epsilon,i}(t)|^2_1 + 1\Big).$$

Integrating over $(0,T)$ and noting (3.2.27) and (3.2.21), it follows that

$$\|w^h_{\epsilon,i}(t)\|_{L^2(0,T;H^1(\Omega))} \ \leq\ C\Big(1 + T^{\frac{1}{2}}\Big).$$

In order to prove uniqueness of a solution to $(\mathbf{P}^h_\epsilon)$, we simply analoue of proving uniqueness of that to $(\mathbf{P}_\epsilon)$. $\square$

Theorem 3.2.2 There exists a unique solution $\{u_1^h, u_2^h, w_1^h, w_2^h\}$ to Problem $(\mathbf{P}^h)$ such that the following stability bounds hold

$$\|u_i^h\|_{H^1(0,T;(H(\Omega))')} \leq C, \tag{3.2.34a}$$

$$\|u_i^h\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \tag{3.2.34b}$$

$$\|w_i^h\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{3.2.34c}$$

Proof: Noting compactness argument from Theorem 3.2.1, there exist subsequences of $u_{\epsilon,i}^h$ and $w_{\epsilon,i}^h$ such that, for $i = 1, 2$,

$$u_{\epsilon,i}^h \rightharpoonup u_i^h \text{ in } H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^1(\Omega)), \tag{3.2.35a}$$

$$u_{\epsilon,i}^h \overset{*}{\rightharpoonup} u_i^h \text{ in } L^\infty(0,T;H^1(\Omega)), \tag{3.2.35b}$$

$$w_{\epsilon,i}^h \rightharpoonup w_i^h \text{ in } L^2(0,T;H^1(\Omega)), \tag{3.2.35c}$$

$$u_{\epsilon,i}^h \to u_i^h \text{ in } L^2(\Omega_T). \tag{3.2.35d}$$

Hence (3.2.34a-c) follow on noting (3.2.35a-c).

We now consider that

$$\begin{aligned}\left|\lim_{\epsilon\to 0} \|u_{\epsilon,i}^h\|_h^2 - \|u_i^h\|_h^2\right| &= \lim_{\epsilon\to 0} \left|\|u_{\epsilon,i}^h\|_h^2 - \|u_i^h\|_h^2\right| = \lim_{\epsilon\to 0} |(u_{\epsilon,i}^h - u_i^h, u_{\epsilon,i}^h + u_i^h)| \\ &\leq \lim_{\epsilon\to 0} \|u_{\epsilon,i}^h - u_i^h\|_h \Big(\|u_{\epsilon,i}^h\|_h + \|u_i^h\|_h\Big).\end{aligned}$$

It follows from (3.2.35d) and convergence properties, see Appendix A, that

$$\lim_{\epsilon\to 0} \|u_{\epsilon,i}^h\|_h^2 = \|u_i^h\|_h^2. \tag{3.2.36}$$

Noting again convergence properties in (3.2.35a-c), we pass to the limit in (3.2.3a,b) to obtain (3.2.1a,b). Setting $\eta = \beta_\epsilon(u_{\epsilon,i}^h) \in H^1(\Omega)$ in (3.2.3b), and using the Cauchy-Schwarz inequality yield

$$\begin{aligned}&\gamma(\nabla u_{\epsilon,i}^h, \nabla \beta_\epsilon(u_{\epsilon,i}^h)) + \frac{1}{\epsilon}|\beta_\epsilon(u_{\epsilon,i}^h)|_h^2 \\ &= (u_{\epsilon,i}^h + w_{\epsilon,i}^h, \beta_\epsilon(u_{\epsilon,i}^h))^h - D(u_{\epsilon,j}^h + 1, \beta_\epsilon(u_{\epsilon,i}^h))^h \\ &\leq \Big(|u_{\epsilon,i}^h|_h + |w_{\epsilon,i}^h|_h\Big)|\beta_\epsilon(u_{\epsilon,i}^h)|_h + D(u_{\epsilon,j}^h + 1, \beta_\epsilon(u_{\epsilon,i}^h))^h \\ &\leq 2\epsilon\Big(|u_{\epsilon,i}^h|_h^2 + |w_{\epsilon,i}^h|_h^2\Big) + \frac{1}{4\epsilon}|\beta_\epsilon(u_{\epsilon,i}^h)|_h^2 + D^2\epsilon|u_{\epsilon,j}^h + 1|_h^2 + \frac{1}{4\epsilon}|\beta_\epsilon(u_{\epsilon,i}^h)|_h^2 \\ &\leq 2\epsilon\Big(|u_{\epsilon,i}^h|_h^2 + |w_{\epsilon,i}^h|_h^2\Big) + \frac{1}{2\epsilon}|\beta_\epsilon(u_{\epsilon,i}^h)|_h^2 + D^2\epsilon\Big(|u_{\epsilon,i}^h|_h^2 + 2m_j + |\Omega|\Big).\end{aligned} \tag{3.2.37}$$

As $0 \leq \beta'_\epsilon \leq 1$, we have

$$\begin{aligned}
(\nabla u^h_{\epsilon,i}, \nabla \beta_\epsilon(u^h_{\epsilon,i})) &= \int_\Omega \beta'_\epsilon(u^h_{\epsilon,i}) \nabla u^h_{\epsilon,i} \nabla u^h_{\epsilon,i}\, dx \\
&\geq \int_\Omega (\beta'_\epsilon(u^h_{\epsilon,i}))^2 \nabla u^h_{\epsilon,i} \nabla u^h_{\epsilon,i}\, dx \\
&= |\beta_\epsilon(u^h_{\epsilon,i})|_1^2 \geq 0, \qquad (3.2.38)
\end{aligned}$$

then it follows on summing with $i = 1, 2$ from (3.2.37) that

$$\sum_{i=1}^{2} \|\beta_\epsilon(u^h_{\epsilon,i})\|^2_{L^2(\Omega_T)} \leq C\epsilon^2. \qquad (3.2.39)$$

Hence, from (3.2.37) and (3.2.38), we obtain

$$\|\beta_\epsilon(u^h_{\epsilon,i})\|_{L^2(0,T;H^1(\Omega))} \leq C\epsilon^{\frac{1}{2}}.$$

If we let $\epsilon \to 0$ then, by (3.2.39), for $a.e. t \in (0, T)$, we have

$$\lim_{\epsilon \to 0} |\beta_\epsilon(u^h_{\epsilon,i})|_0 = 0.$$

Using (2.2.58), (3.2.39) and the Lipschitz continuity of β gives

$$\begin{aligned}
\int_0^T |(\beta(u^h_i), \eta)^h| dt &\leq \int_0^T \Big(|\beta(u^h_i) - \beta(u^h_{\epsilon,i})|_h + |\beta(u^h_{\epsilon,i}) - \beta_\epsilon(u^h_{\epsilon,i})|_h |\beta_\epsilon(u^h_{\epsilon,i})|_h \Big) |\eta|_h\, dt, \\
&\leq C\Big(|u^h_i - u^h_{\epsilon,i}|_{L^2(\Omega_T)} + \epsilon \Big) |\eta|_{L^2(\Omega_T)},
\end{aligned}$$

from (3.2.35d) we have that $\beta(u^h_i) = 0$ $a.e.$ that is $u^h_i \in K^h$.

Let $v \in K^h$ hence $\beta_\epsilon(v) = 0$ and using (3.2.3b) and (2.2.6) to obtain

$$\begin{aligned}
\gamma(\nabla u^h_{\epsilon,i}, \nabla v - \nabla u^h_{\epsilon,i}) - (u^h_{\epsilon,i} + w^h_{\epsilon,i}, v - u^h_{\epsilon,i})^h + D(u^h_{\epsilon,j} + 1, v - u^h_{\epsilon,i})^h& \\
= \frac{1}{\epsilon}(\beta_\epsilon(v) - \beta_\epsilon(u^h_{\epsilon,i}), v - u^h_{\epsilon,i})^h &\geq 0.
\end{aligned}$$

Therefore we have that

$$\gamma(\nabla u^h_{\epsilon,i}, \nabla v - \nabla u^h_{\epsilon,i}) - (u^h_{\epsilon,i}, v - u^h_{\epsilon,i})^h + D(u^h_{\epsilon,j} + 1, v - u^h_{\epsilon,i})^h \geq (w^h_{\epsilon,i}, v - u^h_{\epsilon,i})^h. \quad (3.2.40)$$

We now rearrange (3.2.40) as

$$\begin{aligned}
&\gamma(\nabla u^h_{\epsilon,i}, \nabla v) - (u^h_{\epsilon,i}, v)^h + D(u^h_{\epsilon,j} + 1, v)^h - (w^h_{\epsilon,i}, v)^h \\
&\qquad \geq \gamma(\nabla u^h_{\epsilon,i}, \nabla u^h_{\epsilon,i}) - (u^h_{\epsilon,i}, u^h_{\epsilon,i})^h + D(u^h_{\epsilon,j} + 1, u^h_{\epsilon,i})^h - (w^h_{\epsilon,i}, u^h_{\epsilon,i})^h.
\end{aligned}$$

Noting that

$$\liminf_{\epsilon\to 0} \gamma(\nabla u_{\epsilon,i}^h, \nabla u_{\epsilon,i}^h) \geq \gamma(\nabla u_i^h, \nabla u_i^h),$$

(3.2.36), and the convergence properties of $u_{\epsilon,i}^h$ and $w_{\epsilon,i}^h$, we obtain

$$\begin{aligned}\gamma(\nabla u_i^h, \nabla v) - (u_i^h, v)^h + D(u_j^h + 1, v)^h - (w_i^h, v)^h \\ \geq \gamma(\nabla u_i^h, \nabla u_i^h) - (u_i^h), u_i^h)^h + D(u_j^h + 1, u_i^h)^h - (w_i^h, u_i^h)^h. \quad (3.2.41)\end{aligned}$$

(3.2.1b) can be expressed by rearranging (3.2.41) and (3.2.1a) follows from convergence of (3.2.3a). Hence existence of a solution to Problem ($\mathbf{P}^h$) has been proven.

The existence to Problem ($\mathbf{Q}^h$) follows on noting that

$$w_i^h = -\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t} + \frac{1}{|\Omega|}(w_i^h, 1)^h,$$

then substituting this into (3.2.40) and restricting $\eta \in K^h$ to have a fixed mass, *i.e.* $(\eta, 1) = (u_i^0, 1)$ yields existence.

We now consider uniqueness of Problem ($\mathbf{P}^h$). Let $\{u_1^h, u_2^h, w_1^h, w_2^h\}$ and $\{u_1^{h*}, u_2^{h*}, w_1^{h*}, w_*^{h*}\}$ be two different solutions to Problem ($\mathbf{P}^h$), define $\theta_i^{u^h} = u_i^h - u_i^{h*}$ and $\theta_i^{w^h} = w_i^h - w_i^{h*}$ for $i = 1, 2$.

Subtracting (3.2.3b), when $\{u_1^h, u_2^h, w_1^h, w_2^h\}$ is the solution, from (3.2.3b), when $\{u_1^{h*}, u_2^{h*}, w_1^{h*}, w_2^{h*}\}$ is the solution. Then setting $\eta = u_i^h - u_1^{h*}$ when $i = 1$ and $i = 2$, respectively,

$$\gamma(\nabla\theta_i^{u^h}, \nabla\theta_i^{u^h}) - (\theta_i^{u^h}, \theta_i^{u^h})^h + D(\theta_j^{u^h}, \theta_i^{u^h})^h = (\theta_i^{w^h}, \theta_i^{u^h})^h. \quad (3.2.42)$$

Also subtracting (3.2.3a), when $\{u_1^h, u_2^h, w_1^h, w_2^h\}$ is the solution, from (3.2.3a), when $\{u_1^{h*}, u_2^{h*}, w_1^{h*}, w_2^{h*}\}$ is the solution. Setting $\eta = \widehat{\mathcal{G}}_N^h \theta_i^{u^h}$ when $i = 1$ and $i = 2$ to have

$$\left(\frac{\partial\theta_i^{u^h}}{\partial t}, \widehat{\mathcal{G}}_N^h \theta_i^{u^h}\right)^h = -(\nabla\theta_i^{w^h}, \nabla\widehat{\mathcal{G}}_N^h \theta_i^{u^h}),$$

noting (3.1.4), then we obtain

$$(\theta_i^{w^h}, \theta_i^{u^h})^h = -\left(\frac{\partial\theta_i^{u^h}}{\partial t}, \widehat{\mathcal{G}}_N^h \theta_i^{u^h}\right)^h = -\frac{1}{2}\frac{d}{dt}\|\theta_i^{u^h}\|_{-h}^2. \quad (3.2.43)$$

Substituting (3.2.43) into (3.2.42) and adding together to have

$$\frac{1}{2}\frac{d}{dt}\Big(\|\theta_1^{u^h}\|_{-h}^2+\|\theta_2^{u^h}\|_{-h}^2\Big)+\gamma\Big(|\theta_1^{u^h}|_1^2+|\theta_2^{u^h}|_1^2\Big)+2D(\theta_1^{u^h},\theta_2^{u^h})^h \;\le\; |\theta_1^{u^h}|_h^2+|\theta_2^{u^h}|_h^2. \tag{3.2.44}$$

Noting the Cauchy-Schwarz inequality and (3.1.4) gives

$$\begin{aligned}|\theta_1^{u^h}|_h^2+|\theta_2^{u^h}|_h^2 &= (\nabla\theta_1^{u^h},\nabla\widehat{\mathcal{G}}_N^h\theta_1^{u^h})+(\nabla\theta_2^{u^h},\nabla\widehat{\mathcal{G}}_N^h\theta_2^{u^h})\\ &\le |\theta_1^{u^h}|_1|\widehat{\mathcal{G}}_N^h\theta_1^{u^h}|_1+|\theta_2^{u^h}|_1|\widehat{\mathcal{G}}_N^h\theta_2^{u^h}|_1\\ &= |\theta_1^{u^h}|_1\|\theta_1^{u^h}\|_{-h}+|\theta_2^{u^h}|_1\|\theta_2^{u^h}\|_{-h},\end{aligned} \tag{3.2.45}$$

and

$$\begin{aligned}|2D(\theta_1^{u^h},\theta_2^{u^h})| \;\le\; 2D|\theta_1^{u^h}|_h|\theta_2^{u^h}|_h \;&\le\; D\Big(|\theta_1^{u^h}|_h^2+|\theta_2^{u^h}|_h^2\Big)\\ &\le\; D\Big(|\theta_1^{u^h}|_1\|\theta_1^{u^h}\|_{-h}+|\theta_2^{u^h}|_1\|\theta_2^{u^h}\|_{-h}\Big).\end{aligned} \tag{3.2.46}$$

Noting (3.2.45), (3.2.46) and the Young's inequality (2.1.8) gives

$$\begin{aligned}&(1+D)\Big(|\theta_1^{u^h}|_1\|\theta_1^{u^h}\|_{-h}+|\theta_2^{u^h}|_1\|\theta_2^{u^h}\|_{-h}\Big)\\ &\qquad\le\; \frac{\gamma}{2}\Big(|\theta_1^{u^h}|_1^2+|\theta_2^{u^h}|_1^2\Big)+\frac{(1+D)^2}{2\gamma}\Big(\|\theta_1^{u^h}\|_{-h}^2+\|\theta_2^{u^h}\|_{-h}^2\Big).\end{aligned}$$

Then (3.2.44) becomes

$$\frac{1}{2}\frac{d}{dt}\Big(\|\theta_1^{u^h}\|_{-h}^2+\|\theta_2^{u^h}\|_{-h}^2\Big)+\frac{\gamma}{2}\Big(|\theta_1^{u^h}|_1^2+|\theta_2^{u^h}|_1^2\Big) \;\le\; \frac{(1+D)^2}{2\gamma}\Big(\|\theta_1^{u^h}\|_{-h}^2+\|\theta_2^{u^h}\|_{-h}^2\Big).$$

Using the Grönwall 's inequality to obtain

$$\begin{aligned}&\exp\left(-\frac{-(1+D)^2}{\gamma}(t-0)\right)\Big(\|\theta_1^{u^h}\|_{-h}^2+\|\theta_2^{u^h}\|_{-h}\Big)\\ &\qquad+\gamma\int_0^t\exp\left(\frac{-(1+D)^2s}{\gamma}\right)\Big(|\theta_1^{u^h}(s)|_1^2+|\theta_2^{u^h}(s)|_1^2\Big)\,ds\\ &\qquad\qquad\le \|\theta_1^{u^h}(0)\|_{-h}^2+\|\theta_2^{u^h}(0)\|_{-h}^2 \;=\; 0.\end{aligned} \tag{3.2.47}$$

Noting that $(\theta_i^u,1)^h=0$ and using the discrete Poincaré inequality (3.1.8), it follows that

$$|\theta_i^{u^h}|_h \;\le\; C_p\Big(|\theta_i^{u^h}|_1+|(\theta_i^{u^h},1)^h|\Big) \;\le\; 0.$$

Therefore, we prove the uniqueness of $\{u_1^h,u_2^h\}$. Noting (3.2.5) we also obtian the uniqueness of $\{w_1^h,w_2^h\}$ up to the addition of a constant. Now we have proved the existence and uniqeness to the Problem $(\mathbf{P}^h)$. □

3.3 Error Analysis

Theorem 3.3.1 Let the assumptions of Theorem 3.2.1 hold. Then for all $h > 0$, we have that

$$\|u_1 - u_1^h\|_{L^2(0,T;H^1(\Omega))} + \|u_2 - u_2^h\|_{L^2(0,T;H^1(\Omega))} + \|u_1 - u_1^h\|_{L^\infty(0,T;(H^1(\Omega))')} + \|u_2 - u_2^h\|_{L^\infty(0,T;(H^1(\Omega))')} \le Ch. \tag{3.3.1}$$

Proof: We define

$$e_i = u_i - u_i^h \in V^h,\ e_i^A = u_i - \pi^h u_i \text{ and } e_i^h = \pi^h u_i - u_i^h \in S^h \tag{3.3.2}$$

also note for later analysis that

$$\begin{aligned}(1, e_i) &= (1, u_i(t) - u_i^h(t)) \\ &= (1, u_i^0 - P^h u_i^0) \\ &= (1, u_i^0) - (1, P^h u_i^0)^h \\ &= (1, u_i^0) - (1, u_i^0) = 0,\end{aligned} \tag{3.3.3}$$

$$\begin{aligned}(\eta, e_i^A)^h &= \int_\Omega \pi^h(\eta(u_i - \pi^h u_i))\, dx \\ &= \sum_{j=1}^N \sum_{j=1}^N M_{jj}\eta(x_j)\Big(u_i(x_j) - u_i(x_j)\Big) \\ &= 0 \quad \forall \eta \in H^1(\Omega).\end{aligned} \tag{3.3.4}$$

Choosing $\eta = u_i^h$ in (2.2.2a) gives

$$\gamma(\nabla u_i, \nabla(u_i^h - u_i)) + \left(\mathcal{G}_N \frac{\partial u_i}{\partial t}, u_i^h - u_i\right) - (u_i, u_i^h - u_i) + D(u_j + 1, u_i^h - u_i) \ge 0, \tag{3.3.5}$$

and choosing $\eta = \pi^h u_i$ in (3.2.2a) gives

$$\gamma(\nabla u_i^h, \nabla(\pi^h u_i - u_i^h)) + \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, \pi^h u_i - u_i^h\right)^h - (u_i^h, \pi^h u_i - u_i^h)^h + D(u_j^h + 1, \pi^h u_i - u_i^h)^h \ge 0. \tag{3.3.6}$$

Nothing (3.3.2), (3.3.5) and (3.3.6) become, respectively,

$$\gamma(\nabla u_i, \nabla e_i) + \left(\mathcal{G}_N \frac{\partial u_i}{\partial t}, e_i\right) - (u_i, e_i) + D(u_j + 1, e_i) \le 0, \tag{3.3.7}$$

and

$$\gamma(\nabla u_i^h, \nabla(e_i^A - e_i)) + \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i^A - e_i\right)^h + (u_i^h, e_i^h)^h + D(u_j^h + 1, e_i^h)^h \leq 0. \quad (3.3.8)$$

Adding (3.3.8) to (3.3.7), then subtracting $\left(\mathcal{G}_N \frac{\partial u_i^h}{\partial t}, e_i\right)$ from both sides, we obtain

$$\begin{aligned}
\gamma(\nabla e_i, \nabla e_i) &+ \left(\mathcal{G}_N \frac{\partial u_i}{\partial t}, e_i\right) - \left(\mathcal{G}_N \frac{\partial u_i^h}{\partial t}, e_i\right) \\
&\leq \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i\right)^h - \left(\mathcal{G}_N \frac{\partial u_i^h}{\partial t}, e_i\right) - \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i^A\right)^h \\
&\quad + (u_i, e_i) - (u_i^h, e_i^h)^h \\
&\quad + D(u_j^h + 1, e_i^h)^h - D(u_j + 1, e_i) - \gamma(\nabla u_i^h, \nabla e_i^A). \qquad (3.3.9)
\end{aligned}$$

Noting (3.3.2) and (3.3.4), the third term in the right-hand side of (3.3.9) becomes

$$\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i^A\right)^h = \int_\Omega \pi^h \left(\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}\right)(u_i - \pi^h u_i)\right) dx = 0. \qquad (3.3.10)$$

Considering the first two terms, we then have

$$\begin{aligned}
&\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i\right)^h - \left(\mathcal{G}_N \frac{\partial u_i^h}{\partial t}, e_i\right) \\
&= \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i\right)^h - \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i\right) + \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i\right) - \left(\mathcal{G}_N \frac{\partial u_i^h}{\partial t}, e_i\right). \qquad (3.3.11)
\end{aligned}$$

Noting the Cauchy-Schwarz inequality, (2.1.1a), (3.1.6d) and the Young's inequality (2.1.8), we obtain

$$\begin{aligned}
\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i\right) - \left(\mathcal{G}_N \frac{\partial u_i^h}{\partial t}, e_i\right) &\leq \left|\left((\widehat{\mathcal{G}}_N^h - \mathcal{G}_N)\frac{\partial u_i^h}{\partial t}, e_i\right)\right| \\
&= \left|\left(\nabla(\widehat{\mathcal{G}}_N^h - \mathcal{G}_N)\frac{\partial u_i^h}{\partial t}, \nabla \mathcal{G}_N e_i\right)\right| \\
&\leq \left|(\widehat{\mathcal{G}}_N^h - \mathcal{G}_N)\frac{\partial u_i^h}{\partial t}\right|_1 |\mathcal{G}_N e_i|_1 \\
&\leq Ch\left|\frac{\partial u_i^h}{\partial t}\right|_0 \|e_i\|_{-1} \\
&\leq Ch^2\left|\frac{\partial u_i^h}{\partial t}\right|_0^2 + C\|e_i\|_{-1}^2. \qquad (3.3.12)
\end{aligned}$$

Considering the terms

$$\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i\right)^h - \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i\right) = I_1 + I_2; \qquad (3.3.13)$$

where

$$\begin{aligned} I_1 &= \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, \pi^h e_i\right)^h - \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, \pi^h e_i\right), \\ I_2 &= \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, \pi^h e_i\right) - \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i\right) \\ &= \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, \pi^h e_i - e_i\right). \end{aligned}$$

Noting (3.1.6a) with $r = 1$, (3.3.2), (3.1.6c) with $r = 1$ and $p_2 = 2$ and the Poincaré inequality (2.1.3) gives

$$\begin{aligned} I_1 \le \left|\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, \pi^h e_i\right)^h - \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, \pi^h e_i\right)\right| &\le Ch^2 \left|\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}\right|_1 |\pi^h e_i|_1 \\ &\le Ch^2 \left\|\frac{\partial u_i^h}{\partial t}\right\|_{-1} \left(\|\pi^h e_i - e_i\|_1 + \|e_i\|_1\right) \\ &= Ch^2 \left\|\frac{\partial u_i^h}{\partial t}\right\|_{-1} \left(\|\pi^h u_i - u_i\|_1 + \|e_i\|_1\right) \\ &\le Ch^2 \left\|\frac{\partial u_i^h}{\partial t}\right\|_{-1} \left(Ch|u_i|_2 + C|e_i|_1\right). \end{aligned} \tag{3.3.14}$$

We use the Cauchy-Schwarz inequality, the Poincaré inequality (2.1.3), (3.1.3b), (3.1.4), (3.1.11) and (3.1.6c) to obtain

$$\begin{aligned} I_2 \le \left|\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, \pi^h e_i - e_i\right)\right| &\le \left|\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}\right|_0 |\pi^h e_i - e_i|_0 \\ &\le C \left|\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}\right|_1 |\pi^h u_i - u_i|_0 \\ &\le Ch^2 \left\|\frac{\partial u_i^h}{\partial t}\right\|_{-1} |u_i|_2. \end{aligned} \tag{3.3.15}$$

Substituting (3.3.14) and (3.3.15) into (3.3.13), then substituting (3.3.12) and (3.3.13) into (3.3.11), and noting (3.2.6a), (3.3.11) becomes

$$\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, e_i\right)^h - \left(\mathcal{G}_N \frac{\partial u_i^h}{\partial t}, e_i\right) \le Ch^2 + C|e_i|_1^2 + C\|e_i\|_{-1}^2. \tag{3.3.16}$$

Now we consider the fourth to seventh terms in right-hand side of (3.3.9). Then we note (3.3.2), (3.3.4), (3.1.6a), (3.1.5), the Young's inequality (2.1.8), (3.1.11) and

(2.2.59b) to have

$$\begin{aligned}
(u_i, e_i) - (u_i^h, e_i^h)^h &= (u_i, e_i) - (u_i, e_i)^h + (u_i, e_i)^h - (u_i^h, e_i^h)^h \\
&= (u_i, e_i) - (u_i, e_i)^h + (u_i, e_i^A + e_i^h)^h - (u_i^h, e_i^h)^h \\
&= (u_i, e_i) - (u_i, e_i)^h + (u_i, e_i^A)^h - (e_i, e_i^h)^h \\
&= (u_i, e_i) - (u_i, e_i)^h + |e_i|_h^2 \\
&\leq Ch^2\|u_i\|_1|e_i|_1 + \|e_i\|_{-h}|e_i|_1 \\
&\leq C\|e_i\|_{-1}^2 + \frac{\gamma}{8}|e_i|_1^2 + Ch^4, \qquad (3.3.17)
\end{aligned}$$

and note also (3.3.4) and (3.3.3) to have

$$\begin{aligned}
&D\Big[(u_j + 1, e_i) - D(u_j^h + 1, e_i^h)^h\Big] \\
&= D\Big[(u_j + 1, e_i) - (u_j + 1, e_i)^h + (u_j + 1, e_i)^h - (u_j^h + 1, e_i^h)^h\Big] \\
&= D\Big[(u_j + 1, e_i) - (u_j + 1, e_i)^h\Big] + D\Big[(u_j + 1, e_i)^h - (u_j^h + 1, e_i)^h\Big] \\
&= D\Big[(u_j + 1, e_i) - (u_j + 1, e_i)^h\Big] + D(e_j, e_i)^h \\
&\leq Ch^2\|u_j + 1\|_1|e_i|_1 + C\|e_j\|_{-h}|e_i|_1 \\
&\leq C\|e_j\|_{-1}^2 + \frac{\gamma}{8}|e_i|_1^2 + Ch^4. \qquad (3.3.18)
\end{aligned}$$

The last term in the right-hand side of (3.3.9) can be expressed as

$$\begin{aligned}
-\gamma(\nabla u_i^h, \nabla e_i^A) &= -\gamma(\nabla u_i^h, \nabla e_i^A) + \gamma(\nabla u_i, \nabla e_i^A) - \gamma(\nabla u_i, \nabla e_i^A) \\
&= -\gamma(\nabla u_i, \nabla e_i^A) + \gamma(\nabla(u_i - u_i^h), \nabla e_i^A). \qquad (3.3.19)
\end{aligned}$$

Integrating on the first term of the right-hand side of (3.3.19) by parts and noting $\frac{\partial u_i}{\partial \nu} = 0$ on $\partial\Omega$, the Cauchy-Schwarz inequality, (3.1.6c), and (2.3.15b), we have

$$\begin{aligned}
-\gamma(\nabla u_i, \nabla e_i^A) &\leq \gamma|\Delta u_i|_0|e_i^A|_0 \\
&= \gamma|u_i|_2|(I - \pi^h)u_i|_0 \\
&\leq \gamma|u_i|_2 Ch^2|u_i|_2 \\
&\leq Ch^2. \qquad (3.3.20)
\end{aligned}$$

Noting the Cauchy-Schwarz inequality, (3.3.2), (3.1.6c), the Young's inequality (2.1.8) and (2.2.59b), the second term of the right-hand side of (3.3.19) becomes

$$\begin{aligned}\gamma(\nabla(u_i - u_i^h), \nabla e_i^A) \;=\; \gamma(\nabla e_i, \nabla e_i^A) &\le \gamma|\nabla e_i|_0|\nabla e_i^A|_0 \\ &= \gamma|e_i|_1|(I-\pi^h)u_i|_1 \\ &\le \frac{\gamma}{8}|e_i|_1^2 + 2\gamma Ch^2|u_i|_1^2 \\ &\le \frac{\gamma}{8}|e_i|_1^2 + Ch^2. \end{aligned} \tag{3.3.21}$$

Thus (3.3.19) can be written as

$$-\gamma(\nabla u_i^h, \nabla e_i^A) \;\le\; \frac{\gamma}{8}|e_i|_1^2 + Ch^2. \tag{3.3.22}$$

Substituting (3.3.10), (3.3.16)-(3.3.18) and (3.3.22) into (3.3.9), also noting (3.3.20) and (3.3.21), (3.3.9) can be expressed as

$$\frac{d}{dt}\|e_i\|_{-1}^2 + \gamma|e_i|_1^2 \;\le\; C\|e_i\|_{-1}^2 + C\|e_j\|_{-1}^2 + Ch^2. \tag{3.3.23}$$

Adding (3.3.23) when $i = 2$ to (3.3.23) when $i = 1$, we obtain

$$\frac{d}{dt}\Big(\|e_1\|_{-1}^2 + \|e_2\|_{-1}^2\Big) + \gamma\Big(|e_1|_1^2 + |e_2|_1^2\Big) \;\le\; C\Big(\|e_1\|_{-1}^2 + \|e_2\|_{-1}^2\Big) + Ch^2. \tag{3.3.24}$$

Integrating (3.3.24) over $t \in (0,T)$, using the Grönwall inequality and then rearranging the terms, we then have

$$\begin{aligned}&\|e_1(T)\|_{-1}^2 + \|e_2(T)\|_{-1}^2 + \gamma\int_0^T \Big(|e_1|_1^2 + |e_2|_1^2\Big)\,ds \\ &\qquad\qquad \le\; C\Big(\|e_1(0)\|_{-1}^2 + \|e_2(0)\|_{-1}^2\Big) + C\int_0^T h^2\,ds.\end{aligned}$$

It follows from (3.1.13) that

$$\begin{aligned}&\|e_1\|_{L^\infty(0,T;(H(\Omega))')}^2 + \|e_2\|_{L^\infty(0,T;(H(\Omega))')}^2 + \|e_1\|_{L^2(0,T;H(\Omega))}^2 + \|e_2\|_{L^2(0,T;H(\Omega))}^2 \\ &\quad\le\; C\Big(\|(I-P^h)u_1^0\|_{-1}^2 + \|(I-P^h)u_2^0\|_{-1}^2\Big) + C(T)h^2 \\ &\quad\le\; Ch^2.\end{aligned}$$

The result (3.3.1) can be obtained by noting (3.3.2). □

Chapter 4

A Fully Discrete Approximation

In this chapter we present a numerical scheme to solve the weak formulation we introduced in Chapter 1. The existence, uniqueness, stability and extra stability for the scheme have been investigated in Section 4.2. In Section 4.3 we discuss error estimate between the solutions of weak formulation and fully discrete approximation.

4.1 Notation

We let the assumptions and results of Chapter 3 apply. In addition, we define $\Delta t = \frac{T}{N}$, $t^n = n\Delta t$, $0 \leq n \leq N$, $J^n = (t^{n-1}, t^n], 1 \leq n \leq N$, and

$$\partial \eta^n := \frac{\eta^n - \eta^{n-1}}{\Delta t}, \quad 1 \leq n \leq N, \tag{4.1.1}$$

for a given sequence $\{\eta^n\}_{n=0}^N$, $T > 0$ and $N \in \mathbb{N}$.

We also introduce the operator $\Theta_1^h : H^1(\Omega) \to S^h$ defined by

$$\gamma(\nabla(I - \Theta_1^h)\eta, \nabla\chi) + ((I - \Theta_1^h)\eta, \chi) = 0 \quad \forall \chi \in S^h, \tag{4.1.2}$$

and note that

$$|(I - \Theta_1^h)u_i^0|_{m,p} \leq Ch^\mu |u_i^0|_2, \tag{4.1.3}$$

$$p \in [2, \infty] \text{ and either } \mu = 2 - m - d\left(\frac{1}{2} - \frac{1}{p}\right) \text{ or } \mu \geq 0 \text{ if } p < \infty,$$

see also Barrett and Blowey [7].

4.2 Existence and Uniqueness

Let K^h and $K^h_{m_i}$ be defined as in Section 3.1 and $S^h_{m_i} := \{\chi_i \in S^h : (\chi_i, 1)^h = (U_i^0, 1)^h = m_i\}$. For $\beta \in [0,1]$ we introduce the following fully discrete approximations to Problem $(\mathbf{P})$ and Problem $(\mathbf{Q})$:

Problem $(\mathbf{P}^h_\beta)$ Given $\{U_1^0, U_2^0\}$, find $\{U_1^n, U_2^n, W_1^n, W_2^n\} \in K^h \times K^h \times S^h \times S^h$, for $n = 1, ..., N$, such that

$$(\partial U_1^n, \chi)^h + (\nabla W_1^n, \nabla \chi) = 0, \quad \forall \chi \in S^h \tag{4.2.1a}$$

$$\begin{aligned}\gamma(\nabla U_1^n, \nabla\chi - \nabla U_1^n) - (U_1^{n-1}, \chi - U_1^n)^h + D(\beta U_2^n + (1-\beta)U_2^{n-1}, \chi - U_1^n)^h \\ \geq (W_1^n, \chi - U_1^n)^h \quad \forall \chi \in K^h,\end{aligned} \tag{4.2.1b}$$

and

$$(\partial U_2^n, \chi)^h + (\nabla W_2^n, \nabla \chi) = 0, \quad \forall \chi \in S^h \tag{4.2.2a}$$

$$\begin{aligned}\gamma(\nabla U_2^n, \nabla\chi - \nabla U_2^n) - (U_2^{n-1}, \chi - U_2^n)^h + D((1-\beta) U_1^n + \beta U_1^{n-1}, \chi - U_2^n)^h \\ \geq (W_2^n, \chi - U_2^n)^h \quad \forall \chi \in K^h.\end{aligned} \tag{4.2.2b}$$

Problem $(\mathbf{Q}^h_\beta)$ Given $\{U_1^0, U_2^0\}$, find $\{U_1^n, U_2^n\} \in K^h_{m_1} \times K^h_{m_2}$ for $n = 1, ..., N$, such that, for all $\xi \in K^h_{m_i}$, $i, j = 1, 2$,

$$\begin{aligned}\gamma(\nabla U_1^n, \nabla\chi_1 - \nabla U_1^n) + \left(\widehat{\mathcal{G}}^h_N\left(\frac{U_1^n - U_1^{n-1}}{\Delta t}\right), \chi_1 - U_1^n\right)^h - (U_1^{n-1}, \chi_1 - U_1^n)^h \\ + D(\beta U_2^n + (1-\beta)U_2^{n-1}, \chi_1 - U_1^n)^h \geq 0 \quad \forall \chi_1 \in K^h_{m_1},\end{aligned} \tag{4.2.3a}$$

and

$$\begin{aligned}\gamma(\nabla U_2^n, \nabla\chi_2 - \nabla U_2^n) + \left(\widehat{\mathcal{G}}^h_N\left(\frac{U_2^n - U_2^{n-1}}{\Delta t}\right), \chi_2 - U_2^n\right)^h - (U_2^{n-1}, \chi_2 - U_2^n)^h \\ + D((1-\beta)U_1^n + \beta U_1^{n-1}, \chi_2 - U_2^n)^h \geq 0 \quad \forall \chi_2 \in K^h_{m_2}.\end{aligned} \tag{4.2.3b}$$

Remark: If we observe that

$$W_i^n = -\widehat{\mathcal{G}}^h_N\left(\frac{U_i^n - U_i^{n-1}}{\Delta t}\right) + \lambda_i^n, \tag{4.2.4}$$

where $\lambda_i^n = \frac{(W_i^n,1)^h}{|\Omega|}$ then immediately we see that if $\{U_1^n, U_2^n, W_1^n, W_2^n\}$ solves Problem $(\mathbf{P}_\beta^h)$ then $\{U_1^n, U_2^n\}$ solves Problem $(\mathbf{Q}_\beta^h)$.

Theorem 4.2.1 Let $\Delta t > 0$ and $\mathcal{E}_\gamma^h(U_1^0, U_2^0) \le C$. Then there exist a unique solution $\{U_1^n, U_2^n, W_1^n, W_2^n\}$ satisfying Problem $(\mathbf{P}_\beta^h)$, (4.2.1a,b) and (4.2.2a,b), such that the following stability bounds hold

$$\max_{m=1,\dots,N}\left\{\mathcal{E}_\gamma^h(U_1^m, U_2^m) + \frac{\gamma}{2}\left[\sum_{n=1}^{m}(|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2)\right]\right.$$
$$+\frac{1}{2}\left[\sum_{n=1}^{m}(|U_1^n - U_1^{n-1}|_h^2 + |U_2^n - U_2^{n-1}|_h^2)\right]$$
$$\left.+\Delta t\left[\sum_{n=1}^{m}\left(\left\|\frac{U_1^n - U_1^{n-1}}{\Delta t}\right\|_{-h}^2 + \left\|\frac{U_2^n - U_2^{n-1}}{\Delta t}\right\|_{-h}^2\right)\right]\right\} \le C, \tag{4.2.5}$$

$$\Delta t \sum_{n=1}^{N} \|W_1^n\|_1^2 + \Delta t \sum_{n=1}^{N} \|W_2^n\|_1^2 \le C, \tag{4.2.6}$$

where $\mathcal{E}_\gamma^h(U_1^n, U_2^n) := \frac{\gamma}{2}\left[|U_1^n|_1 + |U_2^n|_1\right] - \frac{1}{2}\left(|U_1^n|_h^2 + |U_2^n|_h^2\right) + D(U_1^n, U_2^n)^h$.

Furthermore, for $\Delta t < \Delta t^*$ where

$$\Delta t^* = \frac{4\gamma}{(\beta_* D)^2}, \tag{4.2.7}$$

and

$$\beta_* = \begin{cases} 1-\beta & \text{if } 0 \le \beta \le \frac{1}{2}, \\ \beta & \text{if } \frac{1}{2} \le \beta \le 1, \end{cases}$$

the solution is uniquely defined.

Proof: We begin by proving stability, i.e. there is a Lyapunov functional similar to the continuous functional (2.2.18d). Substituting $\chi = U_1^{n-1}$ and $\chi = U_2^{n-1}$ into (4.2.1b) and (4.2.2b) respectively to obtain

$$\gamma(\nabla U_1^n, \nabla U_1^{n-1} - \nabla U_1^n) - (U_1^{n-1}, U_1^{n-1} - U_1^n)^h + D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1^{n-1} - U_1^n)^h$$
$$\ge (W_1^n, U_1^{n-1} - U_1^n)^h, \tag{4.2.8}$$

and

$$\gamma(\nabla U_2^n, \nabla U_2^{n-1} - \nabla U_2^n) - (U_2^{n-1}, U_2^{n-1} - U_2^n)^h + D((1-\beta)U_1^n + \beta U_1^{n-1}, U_2^{n-1} - U_2^n)^h$$
$$\ge (W_2^n, U_2^{n-1} - U_2^n)^h, \tag{4.2.9}$$

then noting that $a(a-b) = \frac{1}{2}\left[a^2 + (a-b)^2 - b^2\right]$, we have, for $i = 1, 2$,

$$\gamma(\nabla U_i^n, \nabla U_i^{n-1} - \nabla U_i^n) = -\frac{\gamma}{2}\Big(|U_i^n|_1^2 + |U_i^n - U_i^{n-1}|_1^2 - |U_i^{n-1}|_1^2\Big), \qquad (4.2.10)$$

and

$$(U_i^{n-1}, U_i^{n-1} - U_i^n) = -\frac{1}{2}\Big(|U_i^n|_h^2 - |U_i^n - U_i^{n-1}|_h^2 - |U_i^{n-1}|_h^2\Big). \qquad (4.2.11)$$

Taking $\chi = \widehat{\mathcal{G}}_N^h\left(\frac{U_1^n - U_1^{n-1}}{\Delta t}\right)$ in (4.2.1a) to give

$$\left(\frac{U_1^n - U_1^{n-1}}{\Delta t}, \widehat{\mathcal{G}}_N^h\left(\frac{U_1^n - U_1^{n-1}}{\Delta t}\right)\right)^h = -\left(\nabla W_1^n, \nabla\widehat{\mathcal{G}}_N^h\left(\frac{U_1^n - U_1^{n-1}}{\Delta t}\right)\right). \qquad (4.2.12)$$

Then noting (3.1.4) on the left-hand side and (3.1.3a) on the right-hand side, (4.2.12) becomes

$$\begin{aligned}\left\|\frac{U_1^n - U_1^{n-1}}{\Delta t}\right\|_{-h}^2 &= -\left(W_1^n, \frac{U_1^n - U_1^{n-1}}{\Delta t}\right)^h \\ &= \frac{1}{\Delta t}(W_1^n, U^{n-1} - U_1^n)^h,\end{aligned}$$

thus

$$(W_1^n, U_1^{n-1} - U_1^n)^h = \Delta t\left\|\frac{U_1^n - U_1^{n-1}}{\Delta t}\right\|_{-h}^2. \qquad (4.2.13)$$

Similarly taking $\chi = \widehat{\mathcal{G}}_N^h\left(\frac{U_2^n - U_2^{n-1}}{\Delta t}\right)$ in (4.2.2a). Noting (3.1.4) and (3.1.3a) we also have

$$(W_2^n, U_2^{n-1} - U_2^n)^h = \Delta t\left\|\frac{U_2^n - U_2^{n-1}}{\Delta t}\right\|_{-h}^2. \qquad (4.2.14)$$

Finally, we derive that

$$\begin{aligned}&(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1^{n-1} - U_1^n)^h + ((1-\beta)U_1^n + \beta U_1^{n-1}, U_2^{n-1} - U_2^n)^h \\ &= (\beta U_2^n, U_1^{n-1})^h - (\beta U_2^n, U_1^n)^h + ((1-\beta)U_2^{n-1}, U_1^{n-1})^h - ((1-\beta)U_2^{n-1}, U_1^n)^h \\ &\quad + ((1-\beta)U_1^n, U_2^{n-1})^h - ((1-\beta)U_1^n, U_2^n)^h + (\beta U_1^{n-1}, U_2^{n-1})^h - (\beta U_1^{n-1}, U_2^n)^h \\ &= -(\beta + (1-\beta))(U_1^n, U_2^n)^h + (\beta + (1-\beta))(U_1^{n-1}, U_2^{n-1})^h \\ &= -(U_1^n, U_2^n)^h + (U_1^{n-1}, U_2^{n-1})^h. \end{aligned} \qquad (4.2.15)$$

Substituting (4.2.10), (4.2.11) and (4.2.13) into (4.2.8), and (4.2.10), (4.2.11) and (4.2.14) into (4.2.9) then adding the resulting equations and utilizing (4.2.15) to have

$$
\begin{aligned}
&\mathcal{E}_\gamma^h(U_1^n,U_2^n)+\frac{\gamma}{2}\Big(|U_1^n-U_1^{n-1}|_1^2+|U_2^n-U_2^{n-1}|_1^2\Big)\\
&+\frac{1}{2}\Big(|U_1^n-U_1^{n-1}|_h^2+|U_2^n-U_2^{n-1}|_h^2\Big)+\Delta t\left(\left\|\frac{U_1^n-U_1^{n-1}}{\Delta t}\right\|_{-h}^2+\left\|\frac{U_2^n-U_2^{n-1}}{\Delta t}\right\|_{-h}^2\right)\\
&\le\ \mathcal{E}_\gamma^h(U_1^{n-1},U_2^{n-1}).
\end{aligned}
\tag{4.2.16}
$$

Summing the above equation over $n=1,\dots,m$ for $1\le m\le N$, then noting the Young's inequality (2.1.8) gives

$$
\begin{aligned}
&\max_{m=1,\dots,N}\Bigg\{\mathcal{E}_\gamma^h(U_1^m,U_2^m)+\frac{\gamma}{2}\sum_{n=1}^m\Big(|U_1^{n-1}-U_1^n|_1^2+|U_2^{n-1}-U_2^n|_1^2\Big)\\
&\qquad+\frac{1}{2}\sum_{n=1}^m\Big(|U_1^{n-1}-U_1^n|_h^2+|U_2^{n-1}-U_2^n|_1^2\Big)\\
&\qquad+\Delta t\sum_{n=1}^m\left(\left\|\frac{U_1^n-U_1^{n-1}}{\Delta t}\right\|_{-h}^2+\left\|\frac{U_2^n-U_2^{n-1}}{\Delta t}\right\|_{-h}^2\right)\Bigg\}\\
&\le\ \mathcal{E}_\gamma^h(U_1^0,U_2^0)\\
&\le\ C.
\end{aligned}
\tag{4.2.17}
$$

Noting (4.2.4) and (3.1.4) we have

$$
\begin{aligned}
|\nabla W_i^n|_0 &= \left|\nabla\left(-\widehat{\mathcal{G}}_N^h\left(\frac{U_i^n-U_i^{n-1}}{\Delta t}\right)+\lambda_i\right)\right|_0\\
&= \left(\int_\Omega\left|-\nabla\widehat{\mathcal{G}}_N^h\left(\frac{U_i^n-U_i^{n-1}}{\Delta t}\right)\right|_0^2\right)^{\frac{1}{2}}\\
&= \left|\nabla\widehat{\mathcal{G}}_N^h\left(\frac{U_i^n-U_i^{n-1}}{\Delta t}\right)\right|_0\\
&\equiv \left\|\frac{U_i^n-U_i^{n-1}}{\Delta t}\right\|_{-h},
\end{aligned}
\tag{4.2.18}
$$

then summing from $n=1,\dots,m$ where $1\le m\le N$ and using (4.2.17), (4.2.18) becomes

$$
\Delta t\sum_{n=1}^m|W_1^n|_1^2\ =\ \Delta t\sum_{n=1}^m\left\|\frac{U_1^n-U_1^{n-1}}{\Delta t}\right\|_{-h}^2\ \le\ C. \tag{4.2.19}
$$

Next, consider discrete Poincaré inequality (3.1.8), that is

$$
|W_1^n|_h\ \le\ C_P\Big(|W_1^n|_1+|(W_1^n,1)^h|\Big). \tag{4.2.20}
$$

Setting $\chi = 1$ in (4.2.1b) gives

$$-\gamma|U_1^n|_1^2 - (U_1^{n-1}, 1 - U_1^n)^h + D(\beta U_2^n + (1-\beta)U_2^{n-1}, 1 - U_1^n)^h \geq (W_1^n, 1 - U_1^n)^h, \tag{4.2.21}$$

then

$$-\gamma|U_1^n|_1^2 - m_1 + (U_1^{n-1}, U_1^n)^h + Dm_2 - D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1^n)^h \geq (W_1^n, 1)^h - (W_1^n, U_1^n)^h,$$

and also with $\chi = -1$ in (4.2.1b) gives

$$-\gamma|U_1^n|_1^2 - (U_1^{n-1}, -1 - U_1^n)^h + D(\beta U_2^n + (1-\beta)U_2^{n-1}, -1 - U_1^n)^h \geq (W_1^n, -1 - U_1^n)^h,$$

then

$$-\gamma|U_1^n|_1^2 + m_1 + (U_1^{n-1}, U_1^n)^h - Dm_2 - D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1^n)^h \geq -(W_1^n, 1)^h - (W_1^n, U_1^n)^h.$$

Hence we obtain

$$\begin{aligned}
&\gamma|U_1^n|_1^2 - m_1 - (U_1^{n-1}, U_1^n)^h + Dm_2 + D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1^n)^h - (W_1^n, U_1^n)^h \\
&\leq (W_1^n, 1)^h \\
&\leq -\gamma|U_1^n|_1^2 - m_1 + (U_1^{n-1}, U_1^n)^h + Dm_2 - D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1^n)^h + (W_1^n, U_1^n)^h.
\end{aligned}$$

Therefore, we have that

$$|(W_1^n, 1)^h| \leq |m_1| + |Dm_2| - \gamma|U_1^n|_1^2 + (U_1^{n-1}, U_1^n)^h - D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1^n)^h + (W_1^n, U_1^n)^h. \tag{4.2.22}$$

Using the Cauchy-Schwarz inequality and (4.2.17) to obtain

$$(U_1^{n-1}, U_1^n)^h \leq |U^{n-1}|_h |U^n|_h \leq C,$$

$$D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1^n)^h \leq D|U_1^n|_h \Big(\beta|U_2^n|_h + (1-\beta)|U_2^{n-1}|_h\Big) \leq C,$$

also noting (3.1.3a), (3.1.4) and (3.1.10) to obtain

$$\begin{aligned}
(W_1^n, U_1^n)^h &= \left(W_1^n, U_1^n - \frac{m_1}{|\Omega|}\right)^h + \frac{m_1}{|\Omega|}(W_1^n, 1)^h \\
&= \left(\nabla W_1^n, \nabla \widehat{\mathcal{G}}_N^h \left(U_1^n - \frac{m_1}{|\Omega|}\right)\right) + \frac{m_1}{|\Omega|}(W_1^n, 1)^h \\
&\leq |W_1^n|_1 \left\| U_1^n - \frac{m_1}{|\Omega|} \right\|_{-h} + \frac{|m_1|}{|\Omega|} |(W_1^n, 1)^h| \\
&\leq C_p \left| U_1^n - \frac{m_1}{|\Omega|} \right|_h |W_1^n|_1 + \frac{|m_1|}{|\Omega|} |(W_1^n, 1)^h| \\
&\leq C_0 |W_1^n|_1 + \frac{|m_1|}{|\Omega|} |(W_1^n, 1)^h|,
\end{aligned}$$

noting again (4.2.17), we can rewrite (4.2.22) as

$$|(W_1^n, 1)^h| \leq C + C_0|W_1^n|_1 + \frac{|m_1|}{|\Omega|}\,|(W_1^n, 1)^h|.$$

Now noting that $\frac{|m_1|}{|\Omega|} < 1$, (4.2.19) and (4.2.20), then summing from $n = 1, ..., m$ where $1 \leq m \leq N$ yields

$$\Delta t \sum_{n=1}^{m} \|W_1^n\|_1^2 \leq C.$$

In the same way, setting $\chi = 1, -1$ in (4.2.2b), we then obtain

$$\Delta t \sum_{n=1}^{m} \|W_2^n\|_1^2 \leq C.$$

For uniqueness, let $\{U_1^n, U_2^n\}$ and $\{U_1^{n*}, U_2^{n*}\}$ be two different solutions to Problem $(\mathbf{Q}_\beta^h)$, define $\theta_1^U = U_1^n - U_1^{n*}$ and $\theta_2^U = U_2^n - U_2^{n*}$. By substituting $\chi_1 = U_1^{n*}$ and $\chi_2 = U_2^{n*}$ when $\{U_1^n, U_2^n\}$ is the solution, and vice-versa. Then adding together when $i = 1$ to have

$$\gamma|\theta_1^U|_1^2 + D(\beta\theta_2^U, \theta_1^U)^h + \frac{1}{\Delta t}\|\theta_1^U\|_{-h}^2 \leq 0. \tag{4.2.23}$$

By (3.1.3a) and the Cauchy's inequality (2.1.9), it follows that

$$\begin{aligned}|D(\beta\theta_2^U, \theta_1^U)^h| &= |D\beta(\nabla\widehat{\mathcal{G}}_N^h\theta_1^U, \nabla\theta_2^U)| \\ &\leq \frac{\beta^2}{\delta}\|\theta_1^U\|_{-h}^2 + \frac{\delta D^2}{4}|\theta_2^U|_1^2.\end{aligned} \tag{4.2.24}$$

Thus (4.2.23) becomes

$$\gamma|\theta_1^U|_1^2 + \left(\frac{1}{\Delta t} - \frac{\beta^2}{\delta}\right)\|\theta_1^U\|_{-h}^2 - \frac{\delta D^2}{4}|\theta_2^U|_1^2 \leq 0. \tag{4.2.25}$$

In the same way, when $i = 2$, we also have

$$\gamma|\theta_2^U|_1^2 + \left(\frac{1}{\Delta t} - \frac{(1-\beta)^2}{\delta}\right)\|\theta_2^U\|_{-h}^2 - \frac{\delta D^2}{4}|\theta_1^U|_1^2 \leq 0. \tag{4.2.26}$$

Now, adding (4.2.25) and (4.2.26) gives

$$\begin{aligned}&\left(\gamma - \frac{\delta D^2}{4}\right)\left(|\theta_1^U|_1^2 + |\theta_2^U|_1^2\right) + \frac{1}{\Delta t}\left(\|\theta_1^U\|_{-h}^2 + \|\theta_2^U\|_{-h}^2\right) \\ &\leq \frac{\beta^2}{\delta}\|\theta_1^U\|_{-h}^2 + \frac{(1-\beta)^2}{\delta}\|\theta_2^U\|_{-h}^2 \leq \max\left\{\frac{\beta^2}{\delta}, \frac{(1-\beta)^2}{\delta}\right\}\left(\|\theta_1^U\|_{-h}^2 + \|\theta_2^U\|_{-h}^2\right).\end{aligned} \tag{4.2.27}$$

Let $\beta_* = \max\{\beta, 1-\beta\}$, (4.2.27) becomes

$$\left(\gamma - \frac{\delta D^2}{4}\right)\left(|\theta_1^U|_1^2 + |\theta_2^U|_1^2\right) + \left(\frac{1}{\Delta t} - \frac{\beta_*^2}{\delta}\right)\left(\|\theta_1^U\|_{-h}^2 + \|\theta_2^U\|_{-h}^2\right) \le 0. \qquad (4.2.28)$$

Let $\delta = \beta_*^2 \Delta t$. Then (4.2.28) becomes

$$\left(\gamma - \frac{(\beta_* D)^2 \Delta t}{4}\right)\left(|\theta_1^U|_1^2 + |\theta_2^U|_1^2\right) \le 0.$$

Since $\Delta t < \Delta t^* = \frac{4\gamma}{(\beta_* D)^2}$ and $(\theta_1^U, 1)^h = (\theta_2^U, 1)^h = 0$, the discrete Poincaré inequality (3.1.8), implies that

$$|\theta_1^U|_h^2 = |\theta_2^U|_h^2 = 0, \quad \text{i.e.} \quad \theta_1^U = \theta_2^U \equiv 0,$$

which is contradiction, thus the uniqueness follows.

So far we have stability and uniqueness. To prove existence to Problem $(\mathbf{Q}_\beta^h)$, consider the following minimization problem:

For $1 \le n \le M$ fixed, find $\{U_1, U_2\} \in K_{m_1}^h \times K_{m_2}^h =: K_m^h$ such that

$$\begin{aligned}
\mathcal{I}^h(U_1^n, U_2^n) = \min_{\xi \in K_m^h} \mathcal{I}^h(\xi_1, \xi_2) := & \frac{\gamma}{2}\left(|\xi_1|_1^2 + |\xi_2|_1^2\right) + D\beta(\xi_1, \xi_2)^h \\
& - (\xi_1, U_1^{n-1})^h - (\xi_2, U_2^{n-1})^h \\
& + \frac{1}{2\Delta t}\left(\|\xi_1 - U_1^{n-1}\|_{-h}^2 + \|\xi_2 - U_2^{n-1}\|_{-h}^2\right) \\
& + D(1-\beta)\left[(U_1^{n-1}, \xi_2)^h + (U_2^{n-1}, \xi_1)^h\right].
\end{aligned} \qquad (4.2.29)$$

Assuming that $\{U_1^n, U_2^n\} \in K_{m_1}^h \times K_{m_2}^h$ is the solution of (4.2.29) and let $\{\xi_1, \xi_2\} \in K_{m_1}^h \times K_{m_2}^h$, where $\xi_1 = U_1^n + \lambda(\chi_1 - U_1^n)$ with $\chi_1 \in K_{m_1}^h$, $\xi_2 = U_2^n$ and $0 < \lambda < 1$, then we obtain

$$\begin{aligned}
0 \le\ & \mathcal{I}^h(U_1^n + \lambda(\chi_1 - U_1^n), U_2^n) - \mathcal{I}^h(U_1^n, U_2^n) \\
=\ & \frac{\gamma}{2}\left(|U_1^n + \lambda(\chi_1 - U_1^n)|_1^2 - |U_1^n|_1^2\right) \\
& - (U_1^n + \lambda(\chi_1 - U_1^n), U_1^{n-1})^h + (U_1^n, U_1^{n-1})^h \\
& + D\beta\left[(U_1^n + \lambda(\chi_1 - U_1^n), U_2^n)^h - (U_1^n, U_2^n)^h\right] \\
& + \frac{1}{2\Delta t}\left(\|U_1^n + \lambda(\chi_1 - U_1^n) - U_1^{n-1}\|_{-h}^2 + \|U_1^n - U_1^{n-1}\|_{-h}^2\right) \\
& + D(1-\beta)\left[(U_2^n, U_1^n + \lambda(\chi_1 - U_1^n))^h - (U_2^{n-1}, U_1^n)^h\right].
\end{aligned} \qquad (4.2.30)$$

We have that

$$\begin{aligned}
&\left(|U_1^n + \lambda(\chi_1 - U_1^n)|_1^2 - |U_1^n|_1^2\right)\\
&\qquad = |U_1^n|_1^2 + 2\lambda(\nabla U_1^n, \nabla\chi_1 - \nabla U_1^n) + \lambda^2|\chi_1 - U_1^n|_1^2 - |U_1^n|_1^2\\
&\qquad = 2\lambda(\nabla U_1^n, \nabla\chi_1 - \nabla U_1^n) + \lambda^2|\chi_1 - U_1^n|_1^2, \qquad (4.2.31)
\end{aligned}$$

$$-(U_1^n + \lambda(\chi_1 - U_1^n), U_1^{n-1})^h + (U_1^n, U_1^{n-1})^h = -\lambda(\chi_1 - U_1^n, U_1^{n-1})^h, \qquad (4.2.32)$$

$$D\beta\Big[(U_1^n + \lambda(\chi_1 - U_1^n), U_2^n)^h - (U_1^n, U_2^n)^h\Big] = \lambda D\beta(U_2^n, \chi_1 - U_1^n)^h, \qquad (4.2.33)$$

and

$$D(1-\beta)\Big[(U_2^{n-1}, U_1^n + \lambda(\chi_1 - U_1^n)^h - (U_2^{n-1}, U_1^n)^h\Big] = \lambda D(1-\beta)(U_2^{n-1}, \chi_1 - U_1^n)^h. \qquad (4.2.34)$$

Noting (3.1.3a) and (3.1.4), we obtain

$$\begin{aligned}
&\|U_1^n + \lambda(\chi_1 - U_1^n) - U_1^{n-1}\|_{-h}^2 - \|U_1^n - U_1^{n-1}\|_{-h}^2\\
&= \|U_1^n\|_{-h}^2 + 2\lambda(\nabla\widehat{\mathcal{G}}_N^h U_1^n, \nabla\widehat{\mathcal{G}}_N^h(\chi_1 - U_1^n)) + \lambda^2\|\chi_1 - U_1^n\|_{-h}^2 - 2(\nabla\widehat{\mathcal{G}}_N^h U_1^{n-1}, \nabla\widehat{\mathcal{G}}_N^h U_1^n)\\
&\quad - 2\lambda(\nabla\widehat{\mathcal{G}}_N^h U_1^{n-1}, \nabla\widehat{\mathcal{G}}_N^h(\chi_1 - U_1^n)) + \|U_1^{n-1}\|_{-h}^2 - \|U_1^n\|_{-h}^2 + 2(\nabla\widehat{\mathcal{G}}_N^h U_1^n, \nabla\widehat{\mathcal{G}}_N^h U_1^{n-1})\\
&\quad - \|U_1^{n-1}\|_{-h}^2\\
&= 2\lambda(\nabla\widehat{\mathcal{G}}_N^h(U_1^n - U_1^{n-1}), \nabla\widehat{\mathcal{G}}_N^h(\chi_1 - U_1^n)) + \lambda^2\|\chi_1 - U_1^n\|_{-h}^2\\
&= 2\lambda(\widehat{\mathcal{G}}_N^h(U_1^n - U_1^{n-1}), (\chi_1 - U_1^n)) + \lambda^2\|\chi_1 - U_1^n\|_{-h}^2. \qquad (4.2.35)
\end{aligned}$$

Substituting (4.2.31) - (4.2.35) into (4.2.30), then dividing both sides by λ and letting $\lambda \to 0^+$, we have

$$\begin{aligned}
&\gamma(\nabla U_1^n, \nabla\chi_1 - \nabla U_1^n) - (U_1^{n-1}, \chi_1 - U_1^n)^h + D(\beta U_2^n + (1-\beta)U_2^{n-1}, \chi_1 - U_1^n)^h\\
&\qquad \geq \left(-\widehat{\mathcal{G}}_N^h\left(\frac{U_1^n - U_1^{n-1}}{\Delta t}\right), \chi_1 - U_1^n\right)^h.
\end{aligned}$$

In the same way, letting $\xi_1 = U_1^n \in K_{m_1}^h$ and $\xi_2 = U_2^n + \lambda(\chi_2 - U_2^n) \in K_{m_2}^h$ where $0 < \lambda < 1$, gives

$$\begin{aligned}
&\gamma(\nabla U_2^n, \nabla\chi_2 - \nabla U_2^n) - (U_2^n, \chi_2 - U_2^n)^h + D(\beta U_1^n + (1-\beta)U_1^{n-1}, \chi_2 - U_2^n)^h\\
&\qquad \geq \left(-\widehat{\mathcal{G}}_N^h\left(\frac{U_2^n - U_2^{n-1}}{\Delta t}\right), \chi_2 - U_2^n\right)^h.
\end{aligned}$$

Thus we obtain the existence of $(\mathbf{Q}^h_\beta)$ when $1-\beta=\beta$ (i.e. $\beta=\frac{1}{2}$).

To prove existence of Problem $(\mathbf{P}^h_\beta)$ consider the problem:

Problem $(\mathbf{S}^h_\mu)$ Given $\mu_i \in [\mu_i^L, \mu_i^R]$ find $U_{\mu_i} \in K^h$ such that $\forall \eta \in K^h$, $i=1,2$

$$(U_{\mu_i}, \eta - U_{\mu_i})^h \;\geq\; (f_i, \eta - U_{\mu_i})^h - (g_i, \eta - U_{\mu_i})^h + \mu_i(1, \eta - U_{\mu_i})^h, \qquad (4.2.36)$$

where, for $i=1,2$, $(g_i,\nu)^h = \gamma(\nabla U_i^n, \nabla \nu)$, U_i^n is the unique solution of Problem $(\mathbf{Q}^h_\beta)$, $f_1 = U_1^n + U_1^{n-1} - \widehat{\mathcal{G}}^h_N\left(\frac{U_1^n - U_1^{n-1}}{\Delta t}\right) - D(\beta U_2^n + (1-\beta)U_2^{n-1})$, and $f_2 = U_2^n + U_2^{n-1} - \widehat{\mathcal{G}}^h_N\left(\frac{U_2^n - U_2^{n-1}}{\Delta t}\right) - D((1-\beta)U_1^{n-1} + \beta U_1^n)$.

We define μ_i^L and μ_i^R later. We use (3.1.2) and then express (4.2.36) as

$$\sum_{k=1}^n M_{kk} U_{\mu_i}(x_k)\Big(\eta_k - U_{\mu_i}(x_k)\Big) \;\geq\; \sum_{k=1}^n M_{kk}\Big(f_i(x_k) - g_i(x_k) + \mu_i\Big)\Big(\eta_k - U_{\mu_i}(x_k)\Big). \qquad (4.2.37)$$

Fixing l and taking

$$\eta_k \;=\; \begin{cases} U_{\mu_i}(x_k) & \text{if } k \neq l, \\ 2U_{\mu_i}(x_l) - sign\,(U_{\mu_i}(x_l)),\; U_{\mu_i}(x_l) - sign\,(U_{\mu_i}(x_l)) \text{ and } sign\,(U_{\mu_i}(x_l)) & \text{if } k = l, \end{cases}$$

in (4.2.37) yields

$$\Big(U_{\mu_i}(x_l) - (f_i(x_l) - g_i(x_l) + \mu_i)\Big)\Big(U_{\mu_i}(x_l) - sign\,(U_{\mu_i}(x_l))\Big) \;\geq\; 0 \quad \forall\, l, \qquad (4.2.38)$$

$$-sign\,(U_{\mu_i}(x_l))\Big(U_{\mu_i}(x_l) - (f_i(x_l) - g_i(x_l) + \mu_i)\Big) \;\geq\; 0 \quad \forall\, l, \qquad (4.2.39)$$

$$\Big(U_{\mu_i}(x_l) - (f_i(x_l) - g_i(x_l) + \mu_i)\Big)\Big(sign\,(U_{\mu_i}(x_l)) - U_{\mu_i}(x_l)\Big) \;\geq\; 0 \quad \forall\, l. \qquad (4.2.40)$$

Set

$$\mu_i^R \;=\; 1 - \min_{k=1,\dots,n} f_i(x_k) + \max_{k=1,\dots,n} g_i(x_k),$$

so that

$$f_i(x_i) - g_i(x_i) + \mu_i^R \;=\; 1 + f_i(x_k) - \min f_i(x_k) - g_i(x_k) + \max g_i(x_k) \geq 1,$$

hence we conclude that, for $\eta(x_k) \le 1$,

$$\Big(1 - (f_i(x_k) - g_i(x_k) + \mu_i^R)\Big)\Big(\eta(x_k) - 1\Big) \ \ge\ 0$$

such that at each point 1 satisfies (4.2.38) - (4.2.40) and thus $U_{\mu_i^R}(x_l) \equiv 1$ for all l. Similarly setting

$$\mu_i^L \ = \ 1 - \max_{k=1,\dots,n} f_i(x_k) + \min_{k=1,\dots,n} g_i(x_k),$$

we then conclude that

$$\Big(1 + (f_i(x_i) - g_i(x_i) + \mu_i)\Big)\Big(\eta(x_i) + 1\Big) \ \ge\ 0, \text{ hence } U_{\mu_{i_k}^L}(x_l) \equiv -1 \text{ for all } l.$$

We also have

$$U_{\mu_i}(x_k) \ = \ \begin{cases} -1 & \text{if } f_i - g_i + \mu_i < -1, \\ f_i(x_k) - g_i(x_k) + \mu_i & \text{if } -1 \le f_i(x_k) - g_i(x_k) + \mu_i \le 1, \\ 1 & \text{if } f_i - g_i + \mu_i > 1, \end{cases} \tag{4.2.41}$$

is the unique solution for Problem $(\mathbf{S}_\mu^h)$ which follows from (4.2.38) and (4.2.40) if $U_{\mu_i}(x_k) \ne sign\, U_{\mu_i}(x_k)$.

We define the mapping $\mathcal{M}^h : [\mu_i^L, \mu_i^R] \to \mathbb{R}$ by

$$\mathcal{M}^h(\mu_i) \ = \ (U_{\mu_i}, 1)^h.$$

Let $\mu_i^1, \mu_i^2 \in [\mu_i^L, \mu_i^R]$ then setting $\mu_i = \mu_i^1, \eta = U_{\mu_i^2}$ and $\mu_i = \mu_i^2, \eta = U_{\mu_i^1}$ in (4.2.37) adding the resulting inequalities and we obtain

$$0 \ \le\ |U_{\mu_i^1} - U_{\mu_i^2}|_h^2 \ \le\ \Big(\mathcal{M}^h(\mu_i^1) - \mathcal{M}^h(\mu_i^2)\Big)(\mu_i^1 - \mu_i^2). \tag{4.2.42}$$

From the Cauchy-Schwarz inequality and (4.2.42) we have

$$\frac{|\mathcal{M}^h(\mu_i^1) - \mathcal{M}^h(\mu_i^2)|^2}{|\Omega|} \ \le\ |U_{\mu_i^1} - U_{\mu_i^2}|_h^2 \ \le\ \Big(\mathcal{M}^h(\mu_i^1) - \mathcal{M}^h(\mu_i^2)\Big)(\mu_i^1 - \mu_i^2),$$

that is $\mathcal{M}^h$ is monotone and continuous. Using (4.2.41) so that $\mathcal{M}^h(\mu_i^R) = |\Omega|$ and $\mathcal{M}(\mu_i^L) = -|\Omega|$. It follows from the Intermediate Value Theorem that there exists $\lambda_i \in [\mu_i^L, \mu_i^R]$ such that $\mathcal{M}^h(\lambda_i) = (U_{\lambda_i}, 1)^h = m_i$. Now setting $\chi_1 = U_{\lambda_1}$ in

(4.2.3a), $\chi_2 = U_{\lambda_2}$ in (4.2.3b) and $\eta = U_i^n$ in (4.2.36) with $\mu_i = \lambda$ and noting that $(U_i^n, 1) = m_i$, adding the resulting inequalities yields

$$|U_i^n - U_{\lambda_i}|_h^2 \;\leq\; 0,$$

i.e. $U_{\lambda_i} = U_i^n$ and defining $W_i^n = \widehat{\mathcal{G}}_N^h(\partial U_i^n) - \lambda_i$ we have proved existence.

Remark: $\beta = 0$ is a special case as the two equations decouple and existence is shown in a similar way as above. □

Theorem 4.2.2 Let the conditions of Theorem 4.2.1 and the assumptions of Theorem 2.3.1 hold, $\beta = 0$ or $\frac{1}{2}$, $\delta t = O(h^r), r \leq 2$ and $|u_i^0 - U_i^0|_1 \leq Ch$. Then there exist a solution $\{U_1^n, U_2^n, W_1^n, W_2^n\}$ satisfying Problem $(\mathbf{P}_\beta^h)$, (4.2.1a,b) and (4.2.2a,b), such that the following extra stability bound holds

$$\max_{m=1,\ldots,N}\left\{\frac{\gamma}{4}\sum_{n=1}^{m}\Delta t\left(\left|\frac{U_1^n - U_1^{n-1}}{\Delta t}\right|_1^2 + \left|\frac{U_2^n - U_2^{n-1}}{\Delta t}\right|_1^2\right)\right.$$
$$\left. + \frac{1}{2}\sum_{n=1}^{m}\left(|W_1^n - W_1^{n-1}|_1^2 + |W_2^n - W_2^{n-1}|_1^2\right) + \frac{1}{2}\left(|W_1^m|_1^2 + |W_2^m|_1^2\right)\right\}$$
$$\leq\; C. \tag{4.2.43}$$

Proof: Introducing $W_i^0 \in S^h$ defined by

$$(W_i^0, \chi)^h \;=\; -\gamma(\Delta u_i^0, \chi) - (U_i^0, \chi)^h + D(U_j^0, \chi)^h. \tag{4.2.44}$$

It follows from $\frac{\partial u_1^0}{\partial \nu} = \frac{\partial u_2^0}{\partial \nu} = 0$ and the assumption in Corollary 2.3.2 that

$$\|W_i^0\|_1 \;\leq\; C.$$

For clarity we consider the cases $\beta = 0$ and $\beta = \frac{1}{2}$ separately.

Setting $\beta = 0$ in (4.2.1b) to have

$$(W_1^n, \chi - U_1^n)^h \;\leq\; \gamma(\nabla U_1^n, \nabla\chi - \nabla U_1^n) - (U_1^{n-1}, \chi - U_1^n)^h + D(U_2^{n-1}, \chi - U_1^n)^h. \tag{4.2.45}$$

Let $Y_i^n = \frac{U_i^n - U_i^{n-1}}{\Delta t}$ for $n \geq 1$. Setting $\chi = U_1^{n-1}$ in (4.2.45) to obtain

$$-\Delta t(W_1^n, Y_1^n)^h \;\leq\; -\Delta t\Big[\gamma(\nabla U_1^n, \nabla Y_1^n) - (U_1^{n-1}, Y_1^n)^h + D(U_2^{n-1}, Y_1^n)^h\Big] \qquad n \geq 1. \tag{4.2.46}$$

Setting $\chi = U_1^{n+1}$ in (4.2.45) we have

$$\Delta t(W_1^{n-1}, Y_1^n)^h \;\leq\; \Delta t\Big[\gamma(\nabla U_1^{n-1}, \nabla Y_1^n) - (U_1^{n-2}, Y_1^n)^h + D(U_2^{n-2}, Y_1^n)^h\Big] \qquad n \geq 2. \tag{4.2.47}$$

Adding (4.2.46) and (4.2.47) for $n \geq 2$ gives

$$(W_1^n - W_1^{n-1}, Y_1^n)^h \;\geq\; \Delta t\Big[\gamma|Y_1^n|_1^2 - (Y_1^{n-1}, Y_1^n)^h + D(Y_2^{n-1}, Y_1^n)^h\Big].$$

A similar argument with $n = 1$, and using (4.2.44) with $\chi = \Delta t Y_1^1$, replacing (4.2.47), yields, on nothing $\frac{\partial u_1^0}{\partial \nu} = 0$,

$$(W_1^1 - W_1^0, Y_1^1)^h \;\geq\; \gamma\Delta t|Y_1^1|_1^2 - \gamma(\nabla(u_1^0 - U_1^0), \nabla Y_1^1),$$

or, for $n \geq 1$, we have

$$(W_1^n - W_1^{n-1}, Y_1^n)^h \;\geq \gamma\Delta t|Y_1^n|_1^2 + \begin{cases} -\gamma(\nabla(u_1^0 - U_1^0), \nabla Y_1^1) & \text{if } n = 1, \\ -\Delta t(Y_1^{n-1}, Y_1^n)^h + D\Delta t(Y_2^{n-1}, Y_1^n)^h & \text{if } n \geq 2. \end{cases} \tag{4.2.48}$$

In the same way, setting $\beta = 0$ in (4.2.2b). Choosing $\chi = U_2^{n-1}$ and also $\chi = U_2^{n+1}$ then adding together, for $n \geq 2$, gives

$$(W_2^n - W_2^{n-1}, Y_2^n)^h \;\geq\; \Delta t\Big[\gamma|Y_2^n|_1^2 - (Y_2^{n-1}, Y_2^n)^h + D(Y_1^n, Y_2^n)^h\Big].$$

For $n = 1$, using (4.2.44) with $i = 2, j = 1, \chi = \Delta t Y_2^1$ and noting $\frac{\partial u_2^0}{\partial \nu} = 0$ to have

$$(W_2^1 - W_2^0, Y_2^1)^h \;\geq\; \gamma\Delta t|Y_2^1|_1^2 + D\Delta t(Y_1^1, Y_2^1)^h - \gamma(\nabla(u_2^0 - U_2^0), \nabla Y_2^1).$$

Or, for $n \geq 1$, we obtain

$$(W_2^n - W_2^{n-1}, Y_2^n)^h \;\geq \gamma\Delta t|Y_2^n|_1^2 + D\Delta t(Y_1^n, Y_2^n)^h + \begin{cases} -\gamma(\nabla(u_2^0 - U_2^0), \nabla Y_2^1) & \text{if } n = 1 \\ -\Delta t(Y_2^{n-1}, Y_2^n)^h & \text{if } n \geq 2. \end{cases} \tag{4.2.49}$$

Combining (4.2.48) and (4.2.49), it follows that

$$\begin{aligned}
&(W_1^n - W_1^{n-1}, Y_1^n)^h + (W_2^n - W_2^{n-1}, Y_2^n)^h \\
&\geq \gamma\Delta t\Big(|Y_1^n|_1^2 + |Y_2^n|_1^2\Big) + D\Delta t(Y_1^n, Y_2^n)^h \\
&\quad + \begin{cases} -\gamma(\nabla(u_1^0 - U_1^0), \nabla Y_1^1) - \gamma(\nabla(u_2^0 - U_2^0), \nabla Y_2^1) & \text{if } n = 1, \\ -\Delta t\Big[(Y_1^{n-1}, Y_1^n)^h + (Y_2^{n-1}, Y_2^n)^h\Big] + D\Delta t(Y_2^{n-1}, Y_1^n)^h & \text{if } n \geq 2. \end{cases}
\end{aligned} \tag{4.2.50}$$

Using (4.2.1a), (4.2.2a) and the identity $a(a-b) = \frac{1}{2}(a^2 + (a-b)^2 - b^2)$, the terms on the left-hand side of (4.2.50) can be expressed as

$$\begin{aligned}
&(W_1^n - W_1^{n-1}, Y_1^n)^h + (W_2^n - W_2^{n-1}, Y_2^n)^h \\
&= -(\nabla W_1^n, \nabla(W_1^n - W_1^{n-1})) - (\nabla W_2^n, \nabla(W_2^n - W_2^{n-1})) \\
&= -\frac{1}{2}\Big[\Big(|W_1^n|_1^2 + |W_2^n|_1^2\Big) + \Big(|W_1^n - W_1^{n-1}|_1^2 + |W_2^n - W_2^{n-1}|_1^2\Big) - \Big(|W_1^{n-1}|_1^2 + |W_2^{n-1}|_1^2\Big)\Big].
\end{aligned} \tag{4.2.51}$$

Noting (3.1.5), the Cauchy-Schwarz inequality and the Young's inequality (2.1.8), we also consider the terms on the right-hand side of (4.2.50), we then have that

$$|\Delta t(Y_i^{n-1}, Y_i^n)^h| \leq \frac{\gamma}{4}\Delta t|Y_i^n|_1^2 + \frac{\Delta t}{\gamma}\|Y_i^{n-1}\|_{-h}^2, \; i = 1, 2, \tag{4.2.52}$$

$$\Delta t|D(Y_1^n, Y_2^n)^h| \leq \frac{\gamma}{4}\Delta t|Y_2^n|_1^2 + \frac{D^2}{\gamma}\Delta t\|Y_1^n\|_{-h}^2, \tag{4.2.53}$$

$$\Delta t|D(Y_2^{n-1}, Y_1^n)^h| \leq \frac{\gamma}{4}\Delta t|Y_1^n|_1^2 + \frac{D^2}{\gamma}\Delta t\|Y_2^{n-1}\|_{-h}^2, \tag{4.2.54}$$

and also noting (4.1.3) gives

$$\gamma(\nabla(u_1^0 - U_1^0), \nabla Y_1^1) \leq \frac{\gamma}{4}\Delta t|Y_1^1|_1^2 + \frac{\gamma}{\Delta t}|u_1^0 - U_1^0|_1^2 \leq \frac{\gamma}{4}\Delta t|Y_1^1|_1^2 + \frac{\gamma C h^2}{\Delta t}, \tag{4.2.55}$$

$$\gamma(\nabla(u_2^0 - U_2^0), \nabla Y_2^1) \leq \frac{\gamma}{4}\Delta t|Y_2^1|_1^2 + \frac{\gamma}{\Delta t}|u_2^0 - U_2^0|_1^2 \leq \frac{\gamma}{4}\Delta t|Y_2^1|_1^2 + \frac{\gamma C h^2}{\Delta t}. \tag{4.2.56}$$

Substituting (4.2.51) - (4.2.56) into (4.2.50), then summing the equation over $n =$

$1,\dots,m$ for $1 \leq m \leq N$, it follows that

$$\max_{m=1,\dots,N}\left\{\frac{3\gamma}{4}\sum_{n=1}^{m}\Delta t\left(\left|\frac{U_1^n-U_1^{n-1}}{\Delta t}\right|_1^2+\left|\frac{U_2^n-U_2^{n-1}}{\Delta t}\right|_1^2\right)\right.$$
$$\left.+\frac{1}{2}\sum_{n=1}^{m}\left(|W_1^n-W_1^{n-1}|_1^2+|W_2^n-W_2^{n-1}|_1^2\right)+\frac{1}{2}\left(|W_1^m|_1^2+|W_2^m|_1^2\right)\right\}$$
$$\leq \max_{m=1,\dots,N}\left\{\frac{1+D^2}{\gamma}\sum_{n=1}^{m}\Delta t\left(\left\|\frac{U_1^n-U_1^{n-1}}{\Delta t}\right\|_{-h}^2+\left\|\frac{U_2^n-U_2^{n-1}}{\Delta t}\right\|_{-h}^2\right)\right\}$$
$$+\frac{1}{2}\left(|W_1^0|_1^2+|W_2^0|_1^2\right)+\frac{\gamma C h^2}{\Delta t}.$$

The bound of the first term on the right-hand side follows from (4.2.5). Noting (4.2.4), (3.1.4) and (4.2.5) yields the bound of the second term and hence we have showed (4.2.43) for $\beta = 0$.

Now considering when $\beta = \frac{1}{2}$ in (4.2.1b), we have

$$(W_1^n,\chi-U_1^n)^h \leq \gamma(\nabla U_1^n,\nabla\chi-\nabla U_1^n)-(U_1^{n-1},\chi-U_1^n)^h+\frac{D}{2}(U_2^n+U_2^{n-1},\chi-U_1^n)^h. \tag{4.2.57}$$

Let $Y_i^n = \frac{U_i^n-U_i^{n-1}}{\Delta t}$. Setting $\chi = U_1^{n-1}$ in (4.2.57) gives

$$-\Delta t(W_1^n,Y_1^n)^h \leq -\Delta t\left[\gamma(\nabla U_1^n,\nabla Y_1^n)-(U_1^{n-1},Y_1^n)^h+\frac{D}{2}(U_2^n+U_2^{n-1},Y_1^n)^h\right]$$
$$n \geq 1. \quad (4.2.58)$$

Setting $\chi = U_1^{n+1}$ in (4.2.57) gives

$$\Delta t(W_1^{n-1},Y_1^n)^h \leq \Delta t\left[\gamma(\nabla U_1^{n-1},\nabla Y_1^n)-(U_1^{n-2},Y_1^n)^h+\frac{D}{2}(U_2^{n-1}+U_2^{n-2},Y_1^n)^h\right]$$
$$n \geq 2. \quad (4.2.59)$$

Adding (4.2.58) and (4.2.59), for $n \geq 2$, we obtain

$$(W_1^n-W_1^{n-1},Y_1^n)^h \geq \Delta t\left[\gamma|Y_1^n|_1^2-(Y_1^{n-1},Y_1^n)^h+\frac{D}{2}(Y_2^n+Y_2^{n-1},Y_1^n)^h\right].$$

With $n = 1$ in (4.2.58) and then using (4.2.44) with $i = 1, j = 2, \chi = \Delta t Y_1^1$, replacing (4.2.59) yields

$$(W_1^1-W_1^0,Y_1^1)^h \geq \gamma\Delta t|Y_1^1|_1^2+\frac{D}{2}\Delta t(Y_2^1,Y_1^1)^h-\gamma(\nabla(u_1^0-U_1^0),\nabla Y_1^1),$$

or, for $n \geq 1$, we have

$$(W_1^n - W_1^{n-1}, Y_1^n)^h \geq \gamma \Delta t |Y_1^n|_1^2 + \frac{D}{2}\Delta t (Y_2^n, Y_1^n)^h + \begin{cases} -\gamma(\nabla(u_1^0 - U_1^0), \nabla Y_1^1) & \text{if } n = 1, \\ -\Delta t (Y_1^{n-1}, Y_1^n)^h + \frac{D}{2}\Delta t (Y_2^{n-1}, Y_1^n)^h & \text{if } n \geq 2. \end{cases} \tag{4.2.60}$$

In the same way, setting $\beta = \frac{1}{2}$ in (4.2.2b). Choosing $\chi = U_2^{n-1}$ and also $\chi = U_2^{n+1}$ then adding together, for $n \geq 2$, gives

$$(W_2^n - W_2^{n-1}, Y_2^n)^h \geq \Delta t \left[\gamma |Y_2^n|_1^2 - (Y_2^{n-1}, Y_2^n)^h + \frac{D}{2}(Y_1^n + Y_1^{n-1}, Y_2^n)^h \right],$$

For $n = 1$, we use (4.2.44) with $i = 2, j = 1, \chi = Y_1^2 \Delta t$ and note $\frac{\partial u_2^0}{\partial \nu} = 0$ to obtain

$$(W_2^1 - W_2^0, Y_2^1)^h \geq \gamma \Delta t |Y_2^1|_1^2 + \frac{D}{2}\Delta t (Y_1^1, Y_2^1)^h - \gamma(\nabla(u_2^0 - U_2^0), \nabla Y_2^1).$$

Or, for $n \geq 1$, we have

$$(W_2^n - W_2^{n-1}, Y_2^n)^h \geq \gamma \Delta t |Y_2^n|_1^2 + \frac{D}{2}\Delta t (Y_1^n, Y_2^n)^h + \begin{cases} -\gamma(\nabla(u_2^0 - U_2^0), \nabla Y_2^1) & \text{if } n = 1, \\ -\Delta t (Y_2^{n-1}, Y_2^n)^h + \frac{D}{2}\Delta t (Y_1^{n-1}, Y_2^n)^h & \text{if } n \geq 2. \end{cases} \tag{4.2.61}$$

Combining (4.2.60) and (4.2.61) to have

$$\begin{aligned} &(W_1^n - W_1^{n-1}, Y_1^n)^h + (W_2^n - W_2^{n-1}, Y_2^n)^h \\ &\geq \gamma \Delta t \Big(|Y_1^n|_1^2 + |Y_2^n|_1^2 \Big) \\ &+ \begin{cases} -\gamma(\nabla(u_1^0 - U_1^0), \nabla Y_1^1) - \gamma(\nabla(u_2^0 - U_2^0), \nabla Y_2^1) & \text{if } n = 1, \\ -\Delta t \Big[(Y_1^{n-1}, Y_1^n)^h + (Y_2^{n-1}, Y_2^n)^h \Big] + \frac{D}{2}\Delta t \Big[(Y_2^{n-1}, Y_1^n)^h + (Y_1^{n-1}, Y_2^n)^h \Big] & \text{if } n \geq 2. \end{cases} \end{aligned} \tag{4.2.62}$$

Noting (3.1.5), the Cauchy-Schwarz inequality and the Young's inequality (2.1.8), we consider the term

$$\Delta t |D(Y_1^{n-1}, Y_2^n)^h| \leq \frac{\gamma}{8}\Delta t |Y_2^n|_1^2 + \frac{2D^2}{\gamma}\Delta t \|Y_1^{n-1}\|_{-h}^2, \tag{4.2.63}$$

and

$$|\Delta t(Y_1^{n-1}, Y_1^n)^h| \leq \frac{\gamma}{8}\Delta t|Y_1^n|_1^2 + \frac{2}{\gamma}\|Y_1^{n-1}\|_{-h}^2. \tag{4.2.64}$$

Substituting (4.2.51), (4.2.52), (4.2.54)-(4.2.56), (4.2.63) and (4.2.64) into (4.2.62), then summing the result over $n = 1, ..., m$ for $1 \leq m \leq N$ gives

$$\begin{aligned}
&\max_{m=1,\dots,N}\left\{\frac{3\gamma}{8}\sum_{n=1}^{m}\Delta t\left(\left|\frac{U_1^n - U_1^{n-1}}{\Delta t}\right|_1^2 + \left|\frac{U_2^n - U_2^{n-1}}{\Delta t}\right|_1^2\right)\right.\\
&\left.+\frac{1}{2}\sum_{n=1}^{m}\left(|W_1^n - W_1^{n-1}|_1^2 + |W_2^n - W_2^{n-1}|_1^2\right) + \frac{1}{2}\left(|W_1^m|_1^2 + |W_2^m|_1^2\right)\right\}\\
&\leq \max_{m=1,\dots,N}\left\{\frac{3+4D^2}{\gamma}\sum_{n=1}^{m}\Delta t\left(\left\|\frac{U_1^n - U_1^{n-1}}{\Delta t}\right\|_{-h}^2 + \left\|\frac{U_2^n - U_2^{n-1}}{\Delta t}\right\|_{-h}^2\right)\right\}\\
&+\frac{1}{2}\left(|W_1^0|_1^2 + |W_2^0|_1^2\right) + \frac{\gamma C h^2}{\Delta t}.
\end{aligned}$$

Again, the bound of the first term on the right-handed side follows from (4.2.5). Noting (4.2.4), (3.1.4) and (4.2.5) yields the bound of the second term and hence (4.2.43) is proved for $\beta = \frac{1}{2}$. □

4.3 Error Analysis

In this section, for $i = 1, 2$, we take $U_i^0 = \Theta_1^h u_i^0$ and assume that the assumptions of Theorem 2.3.1 hold. Noting the triangle inequality, (4.1.3) with $m = 0$ and $p = \infty$ and the assumptions of Theorem 2.3.1, we have that

$$\begin{aligned}
|\Theta_1^h u_i^0|_{L^\infty(\Omega)} &\leq |\Theta_1^h u_i^0 - u_i^0|_{L^\infty(\Omega)} + |u_i^0|_{L^\infty(\Omega)}\\
&\leq Ch^{2-\frac{d}{2}}|u_i^0|_2 + 1 - \delta\\
&\leq 1,
\end{aligned}$$

for h sufficiently small hence, from other properties possessed by Θ_1^h, we note that (3.1.12) is satisfied .

For $\beta = 0$ and $\beta = \frac{1}{2}$, we apply the framework in Nochetto, Savaré, and Verdi [42] to analyze the discretization error in the backward Euler method. Introducing the variable

$$\mu(t) := \frac{t_n - t}{\Delta t}, \qquad t \in (t_{n-1}, t_n],\ n = 1 \to N, \tag{4.3.1}$$

and

$$U_i \;:=\; \frac{t-t_n}{\Delta t}U_i^n + \frac{t-t_n}{\Delta t}U_i^{n-1}, \quad t \in [t_{n-1}, t_n],\, i = 1,2. \tag{4.3.2}$$

It follows from (4.3.1) and (4.3.2) that for *a.e.* $t \in (0,T)$, $i = 1,2$,

$$U_i - U_i^n \;=\; -\Delta t \mu \frac{\partial U_i}{\partial t}, \;\text{ and }\; U_i - U_i^{n-1} \;=\; \Delta t(1-\mu)\frac{\partial U_i}{\partial t}. \tag{4.3.3}$$

Let $\mathcal{J}^h : S^h \to \mathbb{R}$ be defined by

$$\mathcal{J}^h(\chi_1,\chi_2) \;:=\; \frac{\gamma}{2}\Big(|\chi_1|_1^2 + |\chi_2|_1^2\Big) \qquad \forall \chi_1,\chi_2 \in S^h. \tag{4.3.4}$$

We then introduce the residual

$$\begin{aligned}\mathcal{R} \;:=\; & \left(\widehat{\mathcal{G}}_N^h\frac{\partial U_1}{\partial t} - U_1^{n-1} + D(\beta U_2^n + (1-\beta)U_2^{n-1}), U_1 - U_1^n\right)^h \\ & + \left(\widehat{\mathcal{G}}_N^h\frac{\partial U_2}{\partial t} - U_2^{n-1} + D((1-\beta)U_1^n + \beta U_1^{n-1}), U_2 - U_2^n\right)^h \\ & + \Big[\mathcal{J}^h(U_1,U_2) - \mathcal{J}^h(U_1^n,U_2^n)\Big],\end{aligned} \tag{4.3.5}$$

and for $n = 1 \to N$, we also introduce

$$\begin{aligned}\widehat{\mathcal{E}}^n \;=\; & \left(U_1^{n-1} - \widehat{\mathcal{G}}_N^h\Big(\frac{U_1^n - U_1^{n-1}}{\Delta t}\Big) - D(\beta U_2^n + (1-\beta)U_2^{n-1}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\right)^h \\ & + \left(U_2^{n-1} - \widehat{\mathcal{G}}_N^h\Big(\frac{U_2^n - U_2^{n-1}}{\Delta t}\Big) - D((1-\beta)U_1^n + \beta U_1^{n-1}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\right)^h \\ & - \frac{1}{\Delta t}\Big[\mathcal{J}^h(U_1^n,U_2^n) - \mathcal{J}^h(U_1^{n-1},U_2^{n-1})\Big].\end{aligned} \tag{4.3.6}$$

We introduce $\{U_1^{-1}, U_2^{-1}\} \in K_{m_1}^h \times K_{m_2}^h$ such that $\{U_1^0 - U_1^{-1}, U_2^0 - U_2^{-1}\} \in V^h \times V^h$ and

$$\gamma(\nabla U_1^0, \nabla\chi) + \left(\widehat{\mathcal{G}}_N^h\Big(\frac{U_1^0 - U_1^{-1}}{\Delta t}\Big), \chi\right)^h - (U_1^{-1},\chi)^h + D(\beta U_2^0 + (1-\beta)U_2^{-1},\chi)^h \;=\; 0, \tag{4.3.7a}$$

and

$$\gamma(\nabla U_2^0, \nabla\chi) + \left(\widehat{\mathcal{G}}_N^h\Big(\frac{U_2^0 - U_2^{-1}}{\Delta t}\Big), \chi\right)^h - (U_2^{-1},\chi)^h + D((1-\beta)U_1^0 + \beta U_1^{-1},\chi)^h \;=\; 0. \tag{4.3.7b}$$

Existence for all Δt for the case $\beta = 0$ is easy as the two equations decouple. To prove existence of (4.3.7a,b) for Δt sufficiently small, we consider the minimization problem

$$\begin{aligned} I(\chi_1,\chi_2) \;=\; & \frac{1}{2\Delta t}\Big(\|\chi_1 - U_1^0\|_{-h}^2 + \|\chi_2 - U_2^0\|_{-h}^2\Big) + \frac{1}{2}\Big(\|\chi_1\|_h^2 + \|\chi_2\|_h^2\Big) \\ & - D(1-\beta)(\chi_1,\chi_2)^h - (f_1,\chi_1)^h - (f_2,\chi_2)^h, \end{aligned} \tag{4.3.8}$$

where

$$(f_1,\xi)^h = D(\beta U_2^0,\xi)^h + \gamma(\nabla U_1^0,\nabla\xi),$$

and

$$(f_2,\xi)^h = D((1-\beta)U_1^0,\xi)^h + \gamma(\nabla U_2^0,\nabla\xi).$$

Before we bound (4.3.8), we note that

$$(\chi_1,\chi_2)^h = \frac{1}{2}(\chi_1 - U_1^0 + U_1^0,\chi_2)^h + \frac{1}{2}(\chi_1,\chi_2 - U_2^0 + U_2^0)^h. \tag{4.3.9}$$

Then noting the Cauchy-Schwartz inequality, (3.1.11) and the Young's inequality (2.1.8) to obtain

$$\begin{aligned}(\chi_1 - U_1^0,\chi_2)^h &\le \|\chi_1 - U_1^0\|_h\|\chi_2\|_h \\ &\le D\|\chi_1 - U_1^0\|_h^2 + \frac{1}{4D}\|\chi_2\|_h^2 \\ &\le D\frac{C}{h^2}\|\chi_1 - U_1^0\|_{-h}^2 + \frac{1}{4D}\|\chi_2\|_h^2,\end{aligned} \tag{4.3.10}$$

and

$$\begin{aligned}(U_1^0,\chi_2)^h &= (U_1^0,\chi_2 - U_2^0 + U_2^0)^h \\ &\le \frac{1}{4D}\|U_1^0\|_h^2 + D\frac{C}{h^2}\|\chi_2 - U_2^0\|_{-h}^2 + (U_1^0,U_2^0)^h.\end{aligned} \tag{4.3.11}$$

Similarly, we also have

$$(\chi_1,\chi_2 - U_2^0)^h \le D\frac{C}{h^2}\|\chi_2 - U_2^0\|_{-h}^2 + \frac{1}{4D}\|\chi_1\|_h^2, \tag{4.3.12}$$

and

$$(\chi_1,U_2^0)^h \le \frac{1}{4D}\|U_2^0\|_h^2 + D\frac{C}{h^2}\|\chi_1 - U_1^0\|_{-h}^2 + (U_1^0,U_2^0)^h. \tag{4.3.13}$$

Noting the Young's inequality (2.1.8), for $i = 1, 2$, gives

$$(f_i,x_i)^h \le 2\|f_i\|_h^2 + \frac{1}{8}\|\chi_i\|_h^2. \tag{4.3.14}$$

Substituting (4.3.10)-(4.3.13) into (4.3.9) and noting (4.3.14), therefore (4.3.8) can be expressed as, $\beta = 0, \frac{1}{2}$,

$$\begin{aligned}I(\chi_1,\chi_2) \ge&\ \left(\frac{1}{2\Delta t} - \frac{D^2(1-\beta)C}{h^2}\right)\left(\|\chi_1 - U_1^0\|_{-h}^2 + \|\chi_2 - U_2^0\|_{-h}^2\right) \\ &+ \left(\frac{1}{4} + \frac{\beta}{8}\right)\left(\|\chi_1\|_h^2 + \|\chi_2\|_h^2\right) \\ &- \frac{1-\beta}{8}\left(\|U_1^0\|_h^2 + \|U_2^0\|_h^2\right) - D(1-\beta)(U_1^0,U_2^0)^h \\ &- 2\left(\|f_1\|_h^2 - \|f_2\|_h^2\right).\end{aligned}$$

Hence, as $I(\chi_1, \chi_2)$ is bounded below for Δt sufficiently small, we have existence.

Now we consider the minimum of the minimization problem

$$I(U_1^{-1}, U_2^{-1}) = \min_{\chi_i \in K^h_{m_i}} I(\chi_1, \chi_2). \tag{4.3.15}$$

Assuming that $\{U_1^{-1}, U_2^{-1}\}$ is the solution of (4.3.15) and letting $\chi_1 = U_1^{-1} + \lambda\xi$, $\chi_2 = U_2^{-1}$, $0 < \lambda < 1, \lambda \in \mathbb{R}$, we have

$$\begin{aligned} 0 &\le I(\chi_1, \chi_2) - I(U_1^{-1}, U_2^{-1}) \\ &\le \frac{1}{2\Delta t}\left(\|U_1^{-1} + \lambda\xi - U_1^0\|^2_{-h} - \|U_1^{-1} - U_1^0\|^2_{-h}\right) + \frac{1}{2}\left(\|U_1^{-1} + \lambda\xi\|^2_h - \|U_1^{-1}\|^2_h\right) \\ &\quad - \lambda D(1-\beta)(\xi, U_2^{-1})^h - \lambda(f_1, \xi). \end{aligned} \tag{4.3.16}$$

Next we note that

$$\begin{aligned} \|U_1^{-1} + \lambda\xi - U_1^0\|^2_{-h} - \|U_1^{-1} - U_1^0\|^2_{-h} &= (\widehat{\mathcal{G}}^h_N(U_1^{-1} - U_1^0), U_1^{-1} - U_1^0)^h \\ &\quad + 2\lambda(\widehat{\mathcal{G}}^h_N(U_1^{-1} - U_1^0), \xi)^h \\ &\quad + \lambda^2(\widehat{\mathcal{G}}^h_N\xi, \xi)^h \\ &\quad - (\widehat{\mathcal{G}}^h_N(U_1^{-1} - U_1^0), U_1^{-1} - U_1^0)^h \\ &= 2\lambda(\widehat{\mathcal{G}}^h_N(U_1^{-1} - U_1^0), \xi)^h + \lambda^2\|\xi\|^2_{-h}, \end{aligned} \tag{4.3.17}$$

and

$$\|U_1^{-1} + \lambda\xi\|^2_h - \|U_1^{-1}\|^2_h = \|U_1^{-1}\|^2_h + 2\lambda(U_1^{-1}, \xi)^h + \lambda^2\|\xi\|^2_h - \|U_1^{-1}\|^2_h. \tag{4.3.18}$$

Substituting (4.3.17) and (4.3.18) into (4.3.16), and dividing both side by λ. Then letting $\lambda \longrightarrow 0$, we obtain

$$\frac{1}{\Delta t}(\widehat{\mathcal{G}}^h_N(U_1^{-1} - U_1^0), \xi)^h + (U_1^{-1}, \xi)^h - D(1-\beta)(U_2^{-1}, \xi)^h = (f_1, \xi)^h. \tag{4.3.19}$$

Multiplying (4.3.19) by -1 to have

$$\frac{1}{\Delta t}(\widehat{\mathcal{G}}^h_N(U_1^0 - U_1^{-1}), \xi)^h - (U_1^{-1}, \xi)^h + D(1-\beta)(U_2^{-1}, \xi)^h = -(f_1, \xi)^h.$$

Similarly letting $\xi_1 = U_1^{-1}$, $\xi_2 = U_2^{-1} + \lambda\xi$, $0 < \xi < 1$, we also have

$$\frac{1}{\Delta t}(\widehat{\mathcal{G}}^h_N(U_2^0 - U_2^{-1}, \xi)^h - (U_2^{-1}, \xi)^h + D(1-\beta)(U_1^{-1}, \xi)^h = -(f_2, \xi)^h.$$

To prove uniqueness of (4.3.7a,b), let $\{U_1^{-1}, U_2^{-1}\}$ and $\{U_{1*}^{-1}, U_{2*}^{-1}\}$ be two solutions, and define $\theta_1^U = U_1^{-1} - U_{1*}^{-1}$ and $\theta_2^U = U_2^{-1} - U_{2*}^{-1}$. Then choosing $\chi = \theta_1^U$ in (4.3.7a) and $\chi = \theta_2^U$ in (4.3.7b) when $\{U_1^{-1}, U_2^{-1}\}$ is the solution, we obtain

$$\gamma(\nabla U_1^0, \nabla\theta_1^U) + \left(\hat{\mathcal{G}}_N^h\left(\frac{U_1^0 - U_1^{-1}}{\Delta t}\right), \theta_1^U\right)^h - (U_1^{-1}, \theta_1^U)^h + D(\beta U_2^0 + (1-\beta)U_2^{-1}, \theta_1^U)^h = 0, \tag{4.3.20}$$

and

$$\gamma(\nabla U_2^0, \nabla\theta_2^U) + \left(\hat{\mathcal{G}}_N^h\left(\frac{U_2^0 - U_2^{-1}}{\Delta t}\right), \theta_2^U\right)^h - (U_2^{-1}, \theta_2^U)^h + D((1-\beta)U_1^0 + \beta U_1^{-1}, \theta_2^U)^h = 0. \tag{4.3.21}$$

Also choosing $\chi = -\theta_1^U$ in (4.3.7a) and $\chi = \theta_2^U$ in (4.3.7b) when $\{U_{1*}^{-1}, U_{2*}^{-1}\}$ is the solution, we have

$$\begin{aligned} -\gamma(\nabla U_1^0, \nabla\theta_1^U) - \left(\hat{\mathcal{G}}_N^h\left(\frac{U_1^0 - U_{1*}^{-1}}{\Delta t}\right), \theta_1^U\right)^h + (U_{1*}^{-1}, \theta_1^U)^h & \\ - D(\beta U_2^0 + (1-\beta)U_{2*}^{-1}, \theta_1^U)^h &= 0, \end{aligned} \tag{4.3.22}$$

and

$$\begin{aligned} -\gamma(\nabla U_2^0, \nabla\theta_2^U) - \left(\hat{\mathcal{G}}_N^h\left(\frac{U_2^0 - U_{2*}^{-1}}{\Delta t}\right), \theta_2^U\right)^h + (U_{2*}^{-1}, \theta_2^U)^h & \\ - D((1-\beta)U_1^0 + \beta U_{1*}^{-1}, \theta_2^U)^h &= 0. \end{aligned} \tag{4.3.23}$$

Adding (4.3.22) to (4.3.20) and (4.3.23) to (4.3.21), then multiplying both sides by -1 to obtain

$$\frac{1}{\Delta t}\|\theta_1^U\|_{-h}^2 + |\theta_1^U|_h^2 - D(1-\beta)(\theta_2^U, \theta_1^U)^h = 0, \tag{4.3.24}$$

and

$$\frac{1}{\Delta t}\|\theta_2^U\|_{-h}^2 + |\theta_2^U|_h^2 - D\beta(\theta_1^U, \theta_2^U)^h = 0. \tag{4.3.25}$$

When $\beta = 0$ uniqueness of θ_2^U and θ_1^U follows from (4.3.24) and (4.3.25). For $\beta = \frac{1}{2}$, adding (4.3.25) to (4.3.24) gives

$$\frac{1}{\Delta t}\left(\|\theta_1^U\|_{-h}^2 + \|\theta_2^U\|_{-h}^2\right) + \left(|\theta_1^U|_h^2 + |\theta_2^U|_h^2\right) - D(\theta_1^U, \theta_2^U)^h = 0.$$

In the same way as (4.2.24), we have that

$$\begin{aligned} D(\theta_1^U, \theta_2^U)^h &= \frac{1}{2}D(\theta_1^U, \theta_2^U)^h + \frac{1}{2}D(\theta_1^U, \theta_2^U)^h \\ &\le \frac{1}{2\delta_1}\|\theta_1^U\|_{-h}^2 + \frac{D^2\delta_1}{8}|\theta_2^U|_1^2 + \frac{1}{2\delta_1}\|\theta_2^U\|_{-h}^2 + \frac{D^2\delta_1}{8}|\theta_1^U|_1^2 \\ &= \frac{1}{2\delta_1}\left(\|\theta_1^U\|_{-h}^2 + \|\theta_2^U\|_{-h}^2\right) + \frac{D^2\delta_1}{8}\left(|\theta_1^U|_1^2 + |\theta_2^U|_1^2\right). \end{aligned}$$

Then, noting (3.1.11) and choosing $2\delta_1 = \Delta t$ to obtain

$$\left(1 - \frac{D^2 \Delta t}{16h^2} C\right) \left(|\theta_1^U|_h^2 + |\theta_2^U|_h^2\right) \leq 0.$$

Therefore we have uniqueness for Δt sufficiently small.

Now we introduce

$$\begin{aligned}
\widehat{\mathcal{D}}^1 := &\left(U_1^0 - \widehat{\mathcal{G}}_N^h\Big(\frac{U_1^1 - U_1^0}{\Delta t}\Big) - D(\beta U_2^1 + (1-\beta)U_2^0), \frac{U_1^1 - U_1^0}{\Delta t}\right)^h \\
+ &\left(U_2^0 - \widehat{\mathcal{G}}_N^h\Big(\frac{U_2^1 - U_2^0}{\Delta t}\Big) - D((1-\beta)U_1^1 + \beta U_1^0), \frac{U_2^1 - U_2^0}{\Delta t}\right)^h \\
- &\left(U_1^{-1} - \widehat{\mathcal{G}}_N^h\Big(\frac{U_1^0 - U_1^{-1}}{\Delta t}\Big) - D(\beta U_2^0 + (1-\beta)U_2^{-1}), \frac{U_1^1 - U_1^0}{\Delta t}\right)^h \\
- &\left(U_2^{-1} - \widehat{\mathcal{G}}_N^h\Big(\frac{U_2^0 - U_2^{-1}}{\Delta t}\Big) - D((1-\beta)U_1^0 + \beta U_1^{-1}), \frac{U_2^1 - U_2^0}{\Delta t}\right)^h,
\end{aligned} \tag{4.3.26}$$

and for $n = 2 \to N$

$$\begin{aligned}
\widehat{\mathcal{D}}^n := &\left(U_1^{n-1} - \widehat{\mathcal{G}}_N^h\Big(\frac{U_1^n - U_1^{n-1}}{\Delta t}\Big) - D(\beta U_2^n + (1-\beta)U_2^{n-1}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\right)^h \\
+ &\left(U_2^{n-1} - \widehat{\mathcal{G}}_N^h\Big(\frac{U_2^n - U_2^{n-1}}{\Delta t}\Big) - D((1-\beta)U_1^n + \beta U_1^{n-1}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\right)^h \\
- &\left(U_1^{n-2} - \widehat{\mathcal{G}}_N^h\Big(\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t}\Big) - D(\beta U_2^{n-1} + (1-\beta)U_2^{n-2}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\right)^h \\
- &\left(U_2^{n-2} - \widehat{\mathcal{G}}_N^h\Big(\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t}\Big) - D((1-\beta)U_1^{n-1} + \beta U_1^{n-2}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\right)^h.
\end{aligned} \tag{4.3.27}$$

Lemma 4.3.1 For $n = 1 \to N$, where $\widehat{\mathcal{E}}^n$ is defined in (4.3.6) and $\widehat{\mathcal{D}}^n$ is defined in (4.3.26) and (4.3.27), we have,

$$\begin{aligned}
\frac{\gamma}{2}\left(|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\right) &\leq \Delta t \widehat{\mathcal{E}}^n \\
&\leq \Delta t \widehat{\mathcal{D}}^n - \frac{\gamma}{2}\left(|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\right) \\
&\leq \Delta t \widehat{\mathcal{D}}^n.
\end{aligned} \tag{4.3.28}$$

For *a.e.* $t \in (0, T)$ we have that

$$\mathcal{R} \leq \mu \Delta t \widehat{\mathcal{E}} \leq \mu \Delta t \widehat{\mathcal{D}}; \tag{4.3.29}$$

where

$$\widehat{\mathcal{E}}(t) := \widehat{\mathcal{E}}^n, \qquad \widehat{\mathcal{D}}(t) := \widehat{\mathcal{D}}^n, \qquad t \in (t_{n-1}, t_n], \qquad n = 1 \to N.$$

Moreover we have that

$$\sum_{n=1}^{N}(\Delta t)^2\widehat{\mathcal{E}}^n \;\leq\; (\Delta t)^2\sum_{n=1}^{N}\widehat{\mathcal{D}}^n \;\leq\; C(\Delta t)^2. \tag{4.3.30}$$

Proof: Rearranging (4.2.16) and noting (4.2.15) and (3.1.4), we have that

$$\begin{aligned}
&\frac{\gamma}{2}\Big(|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\Big)\\
&\quad\leq \frac{\gamma}{2}\Big(|U_1^{n-1}|_1^2 + |U_2^{n-1}|_1^2\Big) - \frac{\gamma}{2}\Big(|U_1^n|_1^2 + |U_2^n|_1^2\Big) - \frac{1}{2}\Big(|U_1^{n-1}|_h^2 + |U_2^{n-1}|_h^2\Big)\\
&\qquad + \frac{1}{2}\Big(|U_1^n|_h^2 + |U_2^n|_h^2\Big) - \frac{1}{2}\Big(|U_1^n - U_1^{n-1}|_h^2 + |U_2^n - U_2^{n-1}|_h^2\Big)\\
&\qquad + D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1^n - U_1^{n-1})^h + D((1-\beta)U_1^n + \beta U_1^{n-1}, U_2^n - U_2^{n-1})^h\\
&\qquad - \frac{1}{\Delta t}\left[\left(\widehat{\mathcal{G}}_N^h\left(\frac{U_1^n - U_1^{n-1}}{\Delta t}\right), \frac{U_1^n - U_1^{n-1}}{\Delta t}\right)^h + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_2^n - U_2^{n-1}}{\Delta t}\right), \frac{U_2^n - U_2^{n-1}}{\Delta t}\right)^h\right].
\end{aligned} \tag{4.3.31}$$

Noting the equality $\frac{1}{2}(a^2 - b^2) \;=\; a(a-b) - \frac{1}{2}(b-a)^2$, for $i = 1, 2$, we obtain

$$\begin{aligned}
-\frac{1}{2}\Big(|U_i^{n-1}|_h^2 - |U_i^n|_h^2\Big) \;&=\; -(U_i^{n-1}, U_i^{n-1} - U_i^n)^h + \frac{1}{2}|U_i^n - U_i^{n-1}|_h^2\\
&=\; (U_i^{n-1}, U_i^n - U_i^{n-1})^h + \frac{1}{2}|U_i^n - U_i^{n-1}|_h^2.
\end{aligned} \tag{4.3.32}$$

Substituting (4.3.32) into (4.3.31) gives

$$\begin{aligned}
&\frac{\gamma}{2}\Big(|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\Big)\\
&\quad\leq \Delta t\left(U_1^{n-1} - \widehat{\mathcal{G}}_N^h\left(\frac{U_1^n - U_1^{n-1}}{\Delta t}\right) + D(\beta U_2^n + (1-\beta)U_2^{n-1}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\right)^h\\
&\qquad + \Delta t\left(U_2^{n-1} - \widehat{\mathcal{G}}_N^h\left(\frac{U_2^n - U_2^{n-1}}{\Delta t}\right) + D((1-\beta)U_1^n + \beta U_1^{n-1}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\right)^h\\
&\qquad - \frac{\gamma}{2}\Big(|U_1^n|_1^2 + |U_2^n|_1^2 - |U_1^{n-1}|_1^2 + |U_2^{n-1}|_1^2\Big).
\end{aligned} \tag{4.3.33}$$

Alternatively, (4.3.33) can be expressed as

$$\frac{\gamma}{2}\Big(|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\Big) \;\leq\; \Delta t\widehat{\mathcal{E}}^n.$$

Considering, for $n \geq 2$, we have that

$$
\begin{aligned}
&\widehat{\mathcal{E}}^n - \widehat{\mathcal{D}}^n \\
&= \frac{1}{\Delta t}\left(U_1^{n-2} - \widehat{\mathcal{G}}_N^h\left(\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t}\right) - D(\beta U_2^{n-1} + (1-\beta)U_2^{n-2}), U_1^n - U_1^{n-1}\right)^h \\
&+ \frac{1}{\Delta t}\left(U_2^{n-2} - \widehat{\mathcal{G}}_N^h\left(\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t}\right) - D((1-\beta)U_1^{n-1} + \beta)U_1^{n-2}), U_2^n - U_2^{n-1}\right)^h \\
&- \frac{1}{\Delta t}\left[\frac{\gamma}{2}(|U_1^n|_1^2 + |U_2^n|_1^2 - \frac{\gamma}{2}(|U_1^{n-1}|_1^2 + |U_2^{n-1}|_1^2)\right]. \qquad (4.3.34)
\end{aligned}
$$

Noting the equation $\frac{1}{2}(a^2 - b^2) = a(a-b) - \frac{1}{2}(a-b)^2$ to obtain

$$
\begin{aligned}
\frac{\gamma}{2}\left(|U_i^n|_1^2 - |U_i^{n-1}|_1^2\right) &= -\frac{\gamma}{2}\left(|U_i^{n-1}|_1^2 - |U_i^n|_1^2\right) \\
&= \gamma(\nabla U_i^{n-1}, \nabla(U_i^n - U_i^{n-1})) + \frac{\gamma}{2}|U_i^n - U_i^{n-1}|_1^2.
\end{aligned}
$$

Hence (4.3.34) can be written as,

$$
\begin{aligned}
&\widehat{\mathcal{E}}^n - \widehat{\mathcal{D}}^n \\
&= \frac{1}{\Delta t}\left(U_1^{n-1} - \widehat{\mathcal{G}}_N^h\left(\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t}\right) - D(\beta U_2^{n-1} + (1-\beta)U_2^{n-2}), U_1^n - U_1^{n-1}\right)^h \\
&+ \frac{1}{\Delta t}\left(U_2^{n-1} - \widehat{\mathcal{G}}_N^h\left(\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t}\right) - D((1-\beta)U_1^{n-1} + \beta)U_1^{n-2}), U_2^n - U_2^{n-1}\right)^h \\
&- \frac{\gamma}{\Delta t}\left[(\nabla U_1^{n-1}, \nabla U_1^n - \nabla U_1^{n-1}) + (\nabla U_2^{n-1}, \nabla U_2^n - \nabla U_2^{n-1})\right] \\
&- \frac{\gamma}{2\Delta t}\left(|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\right). \qquad (4.3.35)
\end{aligned}
$$

For $n \geq 2$, replacing U_1^n by U_1^{n-1} and choosing $\chi_1 = U_1^n$ in (4.2.3a)

$$
\begin{aligned}
&\gamma(\nabla U_1^{n-1}, \nabla(U_1^n - U_1^{n-1})) + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t}\right), U_1^n - U_1^{n-1}\right)^h \\
&\quad - (U_1^{n-2}, U_1^n - U_1^{n-1})^h + D(\beta U_2^{n-1} + (1-\beta)U_2^{n-2}, U_1^n - U_1^{n-1})^h \geq 0, \quad (4.3.36)
\end{aligned}
$$

and replacing U_2^n by U_2^{n-1} and choosing $\chi_2 = U_2^n$ in (4.2.3b)

$$
\begin{aligned}
&\gamma(\nabla U_2^{n-1}, \nabla(U_2^n - U_2^{n-1})) + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t}\right), U_2^n - U_2^{n-1}\right)^h \\
&\quad - (U_2^{n-2}, U_2^n - U_2^{n-1})^h + D((1-\beta)U_1^{n-1} + \beta U_1^{n-2}, U_2^n - U_2^{n-1})^h \geq 0. \quad (4.3.37)
\end{aligned}
$$

Substituting (4.3.36) and (4.3.37) into (4.3.35), for $n \geq 2$, we have that

$$\widehat{\mathcal{E}}^n - \widehat{\mathcal{D}}^n \leq 0.$$

For $n=1$, choosing $\chi = U_1^1 - U_1^0$ in (4.3.7a) and $\chi = U_2^1 - U_2^0$ in (4.3.7b)

$$\gamma(\nabla U_1^0, \nabla(U_1^1 - U_1^0)) + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_1^0 - U_1^{-1}}{\Delta t}\right), U_1^1 - U_1^0\right)^h$$
$$- (U_1^{-1}, U_1^1 - U_1^0)^h + D(\beta U_2^0 + (1-\beta)U_2^{-1}, U_1^1 - U_1^0)^h = 0,$$

and

$$\gamma(\nabla U_2^0, \nabla(U_2^1 - U_2^0)) + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_2^0 - U_2^{-1}}{\Delta t}\right), U_2^1 - U_2^0\right)^h$$
$$- (U_2^{-1}, U_2^1 - U_2^0)^h + D((1-\beta)U_1^0 + \beta U_1^{-1}, U_2^1 - U_2^0)^h = 0.$$

It follows that, for $n=1$,

$$\widehat{\mathcal{E}}^n - \widehat{\mathcal{D}}^n \le 0 - \frac{\gamma}{2\Delta t}\Big(|U_1^1 - U_1^0|_1^2 + |U_2^1 - U_2^0|_1^2\Big).$$

Therefore we obtain, for $n \ge 1$

$$\Delta t \widehat{\mathcal{E}}^n \le \Delta t \widehat{\mathcal{D}}^n - \frac{\gamma}{2}\Big(|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\Big).$$

Now, for $t \in (t_{n-1}, t_n)$, we consider that

$$\begin{aligned}
\mathcal{R} - \mu\Delta t\widehat{\mathcal{E}} = {} & \left(\widehat{\mathcal{G}}_N^h\left(\frac{\partial U_1^n}{\partial t}\right) - U_1^{n-1} + D(\beta U_2^n + (1-\beta)U_2^{n-1}), U_1 - U_1^n\right)^h \\
& + \left(\widehat{\mathcal{G}}_N^h\left(\frac{\partial U_2^n}{\partial t}\right) - U_2^{n-1} + D((1-\beta)U_1^n + \beta U_1^{n-1}), U_2 - U_2^n\right)^h \\
& + \mathcal{J}^h(U_1, U_2) - \mathcal{J}^h(U_1^n, U_2^n) \\
& - \mu\Delta t\left(U_1^{n-1} - \widehat{\mathcal{G}}_N^h\left(\frac{\partial U_1^n}{\partial t}\right) - D(\beta U_2^n + (1-\beta)U_2^{n-1}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\right)^h \\
& - \mu\Delta t\left(U_2^{n-1} - \widehat{\mathcal{G}}_N^h\left(\frac{\partial U_2^n}{\partial t}\right) - D((1-\beta)U_1^n + \beta U_1^{n-1}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\right)^h \\
& + \mu\Big[\mathcal{J}^h(U_1^n, U_2^n) - \mathcal{J}^h(U_1^{n-1}, U_2^{n-1})\Big]. \qquad (4.3.38)
\end{aligned}$$

Noting (4.3.3), we can rewrite the first, second, fifth, and sixth terms in right-hand side of (4.3.38), in the following way. We have that, for $t \in (t_{n-1}, t_n)$,

$$
\begin{aligned}
&\left(\widehat{\mathcal{G}}_N^h\left(\frac{\partial U_1^n}{\partial t}\right) - U_1^{n-1} + D(\beta U_2^n + (1-\beta)U_2^{n-1}), U_1 - U_1^n\right)^h \\
&+ \left(\widehat{\mathcal{G}}_N^h\left(\frac{\partial U_2^n}{\partial t}\right) - U_2^{n-1} + D((1-\beta)U_1^n + \beta U_1^{n-1}), U_2 - U_2^n\right)^h \\
&- \mu\Delta t\left(U_1^{n-1} - \widehat{\mathcal{G}}_N^h\left(\frac{\partial U_1^n}{\partial t}\right) - D(\beta U_2^n + (1-\beta)U_2^{n-1}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\right)^h \\
&- \mu\Delta t\left(U_2^{n-1} - \widehat{\mathcal{G}}_N^h\left(\frac{\partial U_2^n}{\partial t}\right) - D((1-\beta)U_1^n + \beta U_1^{n-1}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\right)^h \\
= &-\mu\Delta t\left(\widehat{\mathcal{G}}_N^h\left(\frac{\partial U_1^n}{\partial t}\right) - U_1^{n-1} + D(\beta U_2^n + (1-\beta)U_2^{n-1}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\right)^h \\
&- \mu\Delta t\left(\widehat{\mathcal{G}}_N^h\left(\frac{\partial U_2^n}{\partial t}\right) - U_2^{n-1} + D((1-\beta)U_1^n + \beta U_1^{n-1}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\right)^h \\
&+ \mu\Delta t\left(\widehat{\mathcal{G}}_N^h\left(\frac{\partial U_1^n}{\partial t}\right) - U_1^{n-1} + D(\beta U_2^n + (1-\beta)U_2^{n-1}), \frac{U_1^n - U_1^{n-1}}{\Delta t}\right)^h \\
&+ \mu\Delta t\left(\widehat{\mathcal{G}}_N^h\left(\frac{\partial U_2^n}{\partial t}\right) - U_2^{n-1} + D((1-\beta)U_1^n + \beta U_1^{n-1}), \frac{U_2^n - U_2^{n-1}}{\Delta t}\right)^h \\
= &\ 0. \qquad (4.3.39)
\end{aligned}
$$

Noting (4.3.4) and (4.3.3) for $t \in (t_{n-1}, t_n)$ yields

$$
\begin{aligned}
&\mathcal{J}^h(U_1, U_2) - \mathcal{J}^h(U_1^n, U_2^n) \\
&= \frac{\gamma}{2}\left(|U_1|_1^2 + |U_2|_1^2 - |U_1^n|_1^2 - |U_2^n|_1^2\right) \\
&= \frac{\gamma}{2}\Big[(\nabla U_1 - \nabla U_1^n, \nabla U_1 + \nabla U_1^n) + (\nabla U_2 - \nabla U_2^n, \nabla U_2 + \nabla U_2^n)\Big] \\
&= \frac{\gamma}{2}\left[-\mu\Delta t\left(\nabla\frac{\partial U_1}{\partial t}, \nabla U_1 + \nabla U_1^n\right) - \mu\Delta t\left(\nabla\frac{\partial U_2}{\partial t}, \nabla U_2 + \nabla U_2^n\right)\right],
\end{aligned}
$$

and with $t = t_n$

$$
\begin{aligned}
&\frac{1}{\Delta t}\Big[\mathcal{J}^h(U_1^n, U_2^n) - \mathcal{J}^h(U_1^{n-1}, U_2^{n-1})\Big] \\
&\qquad = \frac{\gamma}{2}\left[\left(\nabla\frac{\partial U_1}{\partial t}, \nabla U_1^n + \nabla U_1^{n-1}\right) + \left(\nabla\frac{\partial U_2}{\partial t}, \nabla U_2^n + \nabla U_2^{n-1}\right)\right],
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\mathcal{J}^h(U_1,U_2) - \mathcal{J}^h(U_1^n,U_2^n) + \mu\Delta t\left[\frac{1}{\Delta t}\left(\mathcal{J}^h(U_1^n,U_2^n) - \mathcal{J}^h(U_1^{n-1},U_2^{n-1})\right)\right]\\
&= \frac{\gamma}{2}\mu\Delta t\left[\left(\nabla\frac{\partial U_1}{\partial t},\nabla U_1^n + \nabla U_1^{n-1} - \nabla U_1 - \nabla U_1^n\right)\right.\\
&\qquad\left. + \left(\nabla\frac{\partial U_2}{\partial t},\nabla U_2^n + \nabla U_2^{n-1} - \nabla U_2 - \nabla U_2^n\right)\right]\\
&= -\frac{\gamma}{2}\mu\Delta t\left[\left(\nabla\frac{\partial U_1}{\partial t},(1-\mu)\Delta t\nabla\frac{\partial U_1}{\partial t}\right) + \left(\nabla\frac{\partial U_2}{\partial t},(1-\mu)\Delta t\nabla\frac{\partial U_2}{\partial t}\right)\right]\\
&= -\frac{\gamma}{2}\mu(1-\mu)(\Delta t)^2\left(\left|\frac{\partial U_1}{\partial t}\right|_1^2 + \left|\frac{\partial U_2}{\partial t}\right|_1^2\right)\\
&\le 0.
\end{aligned}
\tag{4.3.40}
$$

Noting (4.3.3) and substituting (4.3.39) and (4.3.40) into (4.3.38), it follows that

$$\mathcal{R} - \mu\Delta t\widehat{\mathcal{E}} \le 0.$$

We now consider

$$\sum_{n=2}^{N}\widehat{\mathcal{D}}^n = I_1 + I_2 + I_3 + I_4; \tag{4.3.41}$$

where

$$
\begin{aligned}
I_1 &= \sum_{n=2}^{N}\Delta t\left(\frac{U_1^{n-1}-U_1^{n-2}}{\Delta t},\frac{U_1^n-U_1^{n-1}}{\Delta t}\right)^h + \sum_{n=2}^{N}\Delta t\left(\frac{U_2^{n-1}-U_2^{n-2}}{\Delta t},\frac{U_2^n-U_2^{n-1}}{\Delta t}\right)^h,\\
I_2 &= \sum_{n=2}^{N}\left(\widehat{\mathcal{G}}_N^h\left(\frac{U_1^{n-1}-U_1^{n-2}}{\Delta t}\right) - \widehat{\mathcal{G}}_N^h\left(\frac{U_1^n-U_1^{n-1}}{\Delta t}\right),\frac{U_1^n-U_1^{n-1}}{\Delta t}\right)^h,\\
&\quad + \sum_{n=2}^{N}\left(\widehat{\mathcal{G}}_N^h\left(\frac{U_2^{n-1}-U_1^{n-2}}{\Delta t}\right) - \widehat{\mathcal{G}}_N^h\left(\frac{U_2^n-U_2^{n-1}}{\Delta t}\right),\frac{U_2^n-U_2^{n-1}}{\Delta t}\right)^h,\\
I_3 &= -\sum_{n=2}^{N}\Delta tD\left(\beta\left(\frac{U_2^n-U_2^{n-1}}{\Delta t}\right) + (1-\beta)\left(\frac{U_2^{n-1}-U_2^{n-2}}{\Delta t}\right),\frac{U_1^n-U_1^{n-1}}{\Delta t}\right)^h,\\
I_4 &= -\sum_{n=2}^{N}\Delta tD\left((1-\beta)\left(\frac{U_1^n-U_1^{n-1}}{\Delta t}\right) + \beta\left(\frac{U_1^{n-1}-U_1^{n-2}}{\Delta t}\right),\frac{U_2^n-U_2^{n-1}}{\Delta t}\right)^h.
\end{aligned}
$$

Noting (3.1.4), (4.2.5) and the equation $a(a-b) = -\frac{1}{2}[a^2 - b^2 + (a-b)^2]$ yields

$$
\begin{aligned}
I_2 &= \sum_{n=2}^{N} \left(\nabla \widehat{\mathcal{G}}_N^h \left(\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t} \right) - \nabla \widehat{\mathcal{G}}_N^h \left(\frac{U_1^n - U_1^{n-1}}{\Delta t} \right), \nabla \widehat{\mathcal{G}}_N^h \left(\frac{U_1^n - U_1^{n-1}}{\Delta t} \right) \right)^h \\
&\quad + \sum_{n=2}^{N} \left(\nabla \widehat{\mathcal{G}}_N^h \left(\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t} \right) - \nabla \widehat{\mathcal{G}}_N^h \left(\frac{U_2^n - U_2^{n-1}}{\Delta t} \right), \nabla \widehat{\mathcal{G}}_N^h \left(\frac{U_2^n - U_2^{n-1}}{\Delta t} \right) \right) \\
&= \sum_{n=2}^{N} \frac{1}{2} \left(- \left\| \frac{U_1^n - U_1^{n-1}}{\Delta t} \right\|_{-h}^2 + \left\| \frac{U_1^{n-1} - U_1^{n-2}}{\Delta t} \right\|_{-h}^2 \right. \\
&\qquad \left. - \left| \nabla \widehat{\mathcal{G}}_N^h \left(\frac{U_1^n - U_1^{n-1}}{\Delta t} \right) - \nabla \widehat{\mathcal{G}}_N^h \left(\frac{U_1^{n-1} - U_1^{n-2}}{\Delta t} \right) \right|_0^2 \right) \\
&\quad + \sum_{n=2}^{N} \frac{1}{2} \left(- \left\| \frac{U_2^n - U_2^{n-1}}{\Delta t} \right\|_{-h}^2 + \left\| \frac{U_2^{n-1} - U_2^{n-2}}{\Delta t} \right\|_{-h}^2 \right. \\
&\qquad \left. - \left| \nabla \widehat{\mathcal{G}}_N^h \left(\frac{U_2^n - U_2^{n-1}}{\Delta t} \right) - \nabla \widehat{\mathcal{G}}_N^h \left(\frac{U_2^{n-1} - U_2^{n-2}}{\Delta t} \right) \right|_0^2 \right) \\
&\le \frac{1}{2} \left(\left\| \frac{U_1^1 - U_1^0}{\Delta t} \right\|_{-h}^2 + \left\| \frac{U_2^1 - U_2^0}{\Delta t} \right\|_{-h}^2 \right) \\
&\le C.
\end{aligned} \tag{4.3.42}
$$

Noting (3.1.5) and the Cauchy-Schwarz inequality to give

$$
\begin{aligned}
I_1 &\le \sum_{n=2}^{N} \Delta t \left\| \frac{U_1^n - U_1^{n-1}}{\Delta t} \right\|_{-h} \left| \frac{U_1^{n-1} - U_1^{n-2}}{\Delta t} \right|_1 \\
&\qquad + \sum_{n=2}^{N} \Delta t \left\| \frac{U_2^n - U_2^{n-1}}{\Delta t} \right\|_{-h} \left| \frac{U_2^{n-1} - U_2^{n-2}}{\Delta t} \right|_1,
\end{aligned} \tag{4.3.43}
$$

$$
\begin{aligned}
|I_3| &\le \sum_{n=2}^{N} D\,\Delta t \left(\beta \left\| \frac{U_2^n - U_2^{n-1}}{\Delta t} \right\|_{-h} \left| \frac{U_1^n - U_1^{n-1}}{\Delta t} \right|_1 \right. \\
&\qquad \left. + (1-\beta) \left\| \frac{U_1^n - U_1^{n-1}}{\Delta t} \right\|_{-h} \left| \frac{U_2^{n-1} - U_2^{n-2}}{\Delta t} \right|_1 \right),
\end{aligned} \tag{4.3.44}
$$

and

$$
\begin{aligned}
|I_4| &\le \sum_{n=2}^{N} D\,\Delta t \left((1-\beta) \left\| \frac{U_2^n - U_2^{n-1}}{\Delta t} \right\|_{-h} \left| \frac{U_1^n - U_1^{n-1}}{\Delta t} \right|_1 \right. \\
&\qquad \left. + \beta \left\| \frac{U_1^n - U_1^{n-1}}{\Delta t} \right\|_{-h} \left| \frac{U_2^{n-1} - U_2^{n-2}}{\Delta t} \right|_1 \right).
\end{aligned} \tag{4.3.45}
$$

Adding (4.3.44) and (4.3.45) to (4.3.43) and noting (4.2.5), we obtain

$$\begin{aligned}
&|I_1| + |I_3| + |I_4| \\
&\leq \sum_{n=2}^{N} \Delta t \left(\left\| \frac{U_1^n - U_1^{n-1}}{\Delta t} \right\|_{-h} + D \left\| \frac{U_2^n - U_2^{n-1}}{\Delta t} \right\|_{-h} \right) \left| \frac{U_1^{n-1} - U_1^{n-2}}{\Delta t} \right|_1 \\
&\quad + \sum_{n=2}^{N} \Delta t \left(\left\| \frac{U_2^n - U_2^{n-1}}{\Delta t} \right\|_0 + D \left\| \frac{U_1^n - U_1^{n-1}}{\Delta t} \right\|_{-h} \right) \left| \frac{U_2^{n-1} - U_2^{n-2}}{\Delta t} \right|_1 \\
&\leq C. \qquad (4.3.46)
\end{aligned}$$

Therefore, substituting (4.3.42) and (4.3.46) into (4.3.41), it follows that

$$\sum_{n=2}^{N} \widehat{\mathcal{D}}^n \leq C.$$

For $n = 1$, choosing $\chi = \frac{U_1^1 - U_1^0}{\Delta t}$ in (4.3.7a) and $\chi = \frac{U_2^1 - U_2^0}{\Delta t}$ in (4.3.7b) gives

$$\begin{aligned}
&\left(U_1^{-1} - \widehat{\mathcal{G}}_N^h \left(\frac{U_1^0 - U_1^{-1}}{\Delta t} \right) - D(\beta U_2^0 + (1-\beta) U_2^{-1}), \frac{U_1^1 - U_1^0}{\Delta t} \right)^h \\
&\qquad = \gamma \left(\nabla U_1^0, \nabla \left(\frac{U_1^1 - U_1^0}{\Delta t} \right) \right), \qquad (4.3.47)
\end{aligned}$$

and

$$\begin{aligned}
&\left(U_2^{-1} - \widehat{\mathcal{G}}_N^h \left(\frac{U_2^0 - U_2^{-1}}{\Delta t} \right) - D((1-\beta) U_1^0 + \beta U_1^{-1}), \frac{U_2^1 - U_2^0}{\Delta t} \right)^h \\
&\qquad = \gamma \left(\nabla U_2^0, \nabla \left(\frac{U_2^1 - U_2^0}{\Delta t} \right) \right)^h. \qquad (4.3.48)
\end{aligned}$$

Choosing $\chi = \frac{U_1^1 - U_1^0}{\Delta t}$ when $i = 1$ and $\chi = \frac{U_2^1 - U_2^0}{\Delta t}$ when $i = 2$ in (4.2.44), then integrating by parts to have

$$\left(U_1^0, \frac{U_1^1 - U_1^0}{\Delta t} \right)^h = \gamma \left(\nabla u_1^0, \nabla \frac{U_1^1 - U_1^0}{\Delta t} \right) - \left(W_1^0, \frac{U_1^1 - U_1^0}{\Delta t} \right)^h + D \left(U_2^0, \frac{U_1^1 - U_1^0}{\Delta t} \right)^h, \qquad (4.3.49)$$

and

$$\left(U_2^0, \frac{U_2^1 - U_2^0}{\Delta t} \right)^h = \gamma \left(\nabla u_2^0, \nabla \frac{U_2^1 - U_2^0}{\Delta t} \right) - \left(W_2^0, \frac{U_2^1 - U_2^0}{\Delta t} \right)^h + D \left(U_1^0, \frac{U_2^1 - U_2^0}{\Delta t} \right)^h. \qquad (4.3.50)$$

Substituting (4.3.47)-(4.3.50) into (4.3.26), then noting (3.1.5), the Cauchy-Schwarz inequality, (4.1.2), (3.1.12), (4.2.43), (4.2.5) and (4.2.6), it follows that

$$\begin{aligned}
\widehat{\mathcal{D}}^1 &= \gamma\left(\nabla(u_1^0 - U_1^0), \nabla\left(\frac{U_1^1 - U_1^0}{\Delta t}\right)\right) + \gamma\left(\nabla(u_2^0 - U_2^0), \nabla\left(\frac{U_2^1 - U_2^0}{\Delta t}\right)\right) \\
&\quad - \left[\left(W_1^0, \frac{U_1^1 - U_1^0}{\Delta t}\right)^h + \left(W_2^0, \frac{U_2^1 - U_2^0}{\Delta t}\right)^h\right] \\
&\quad - \left[\left(\widehat{\mathcal{G}}_N^h\left(\frac{U_1^1 - U_1^0}{\Delta t}\right), \frac{U_1^1 - U_1^0}{\Delta t}\right)^h + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_2^1 - U_2^0}{\Delta t}\right), \frac{U_2^1 - U_2^0}{\Delta t}\right)^h\right] \\
&\quad - \left[D\beta\left(U_2^1 + U_2^0, \frac{U_1^1 - U_1^0}{\Delta t}\right)^h + D(1-\beta)\left(U_1^1 + U_1^0, \frac{U_2^1 - U_2^0}{\Delta t}\right)^h\right] \\
&\leq \gamma|U_1^0 - u_1^0|_1 \left|\frac{U_1^1 - U_1^0}{\Delta t}\right|_1 + \gamma|U_2^0 - u_2^0|_1 \left|\frac{U_2^1 - U_2^0}{\Delta t}\right|_1 \\
&\quad + |W_1^0|_1 \left|\nabla\widehat{\mathcal{G}}_N^h\left(\frac{U_1^1 - U_1^0}{\Delta t}\right)\right|_0 + |W_2^0|_1 \left|\nabla\widehat{\mathcal{G}}_N^h\left(\frac{U_2^1 - U_2^0}{\Delta t}\right)\right|_0 \\
&\quad + D\beta\left(|U_2^1|_1 \left|\nabla\widehat{\mathcal{G}}_N^h\left(\frac{U_1^1 - U_1^0}{\Delta t}\right)\right|_0 + |U_2^0|_1 \left|\nabla\widehat{\mathcal{G}}_N^h\left(\frac{U_1^1 - U_1^0}{\Delta t}\right)\right|_0\right) \\
&\quad + D(1-\beta)\left(|U_1^1|_1 \left|\nabla\widehat{\mathcal{G}}_N^h\left(\frac{U_2^1 - U_2^0}{\Delta t}\right)\right|_0 + |U_1^0|_1 \left|\nabla\widehat{\mathcal{G}}_N^h\left(\frac{U_2^1 - U_2^0}{\Delta t}\right)\right|_0\right) \\
&= \gamma|U_1^0 - u_1^0|_1 \left|\frac{U_1^1 - U_1^0}{\Delta t}\right|_1 + \gamma|U_2^0 - u_2^0|_1 \left|\frac{U_2^1 - U_2^0}{\Delta t}\right|_1 \\
&\quad + |W_1^0|_1 \left\|\frac{U_1^1 - U_1^0}{\Delta t}\right\|_{-h} + |W_2^0|_1 \left\|\frac{U_2^1 - U_2^0}{\Delta t}\right\|_{-h} \\
&\quad + D\beta\left(|U_2^1|_1 \left\|\frac{U_1^1 - U_1^0}{\Delta t}\right\|_{-h} + |U_2^0|_1 \left\|\frac{U_1^1 - U_1^0}{\Delta t}\right\|_{-h}\right) \\
&\quad + D(1-\beta)\left(|U_1^1|_1 \left\|\frac{U_2^1 - U_2^0}{\Delta t}\right\|_{-h} + |U_1^0|_1 \left\|\frac{U_2^1 - U_2^0}{\Delta t}\right\|_{-h}\right) \\
&\leq C.
\end{aligned}$$

□

Lemma 4.3.2 Let the assumptions of Theorem 2.3.1 hold. Then for *a.e.* $t \in (0,T)$

$$\begin{aligned}
&\frac{\gamma}{2}\left(|E_1|_1^2 + |E_2|_1^2 + |E_1^+|_1^2 + |E_2^+|_1^2\right) + \frac{1}{2}\frac{d}{dt}\left(\|E_1\|_{-h}^2 + \|E_2\|_{-h}^2\right) \\
&\leq (E_1^- - D(\beta E_2^+ + (1-\beta)E_2^-, E_1)^h + (E_2^- - D(1-\beta)E_1^+ + \beta E_1^-, E_2)^h + \mathcal{R},
\end{aligned} \tag{4.3.51}$$

where, for $i = 1, 2$,

$$E_i^{(\pm)} := u_i^h - U_i^{(\pm)} \in V^h \text{ and } U_i^+ = U_i^n,\ U_i^- = U_i^{n-1}. \tag{4.3.52}$$

Proof: Choosing $\eta = U_i \in K^h$, in (3.2.2a) gives

$$\gamma(\nabla u_i^h, \nabla U_i - \nabla u_i^h) - \left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, E_i\right)^h + (u_i^h, E_i)^h - D(u_j^h + 1, E_i)^h \geq 0.$$

Noting the equality $a(b-a) = \frac{1}{2}(-(a-b)^2 + b^2 - a^2)$, we have

$$\frac{\gamma}{2}\left(|E_i|_1^2 - |U_i|_1^2 + |u_i^h|_1^2\right) \leq -\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t} - u_i^h + D(u_j^h + 1), E_i\right)^h, i = 1, 2. \quad (4.3.53)$$

Choosing $\chi_1 = u_1^h$ in (4.2.3a) gives

$$\gamma(\nabla U_1^n, \nabla E_1^+) + \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_1^n}{\partial t}, E_1^+\right)^h - (U_1^{n-1}, E_1^+)^h + D(\beta U_2^n + (1-\beta)U_2^{n-1}, E_1^+)^h \geq 0.$$

Again noting $a(b-a) = \frac{1}{2}(-(a-b)^2 + b^2 - a^2)$, we have

$$\frac{\gamma}{2}\left(|E_1^+|_1^2 + |U_1^n|_1^2 - |u_1^h|_1^2\right) \leq \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_1^n}{\partial t} - U_1^{n-1} + D(\beta U_2^n + (1-\beta)U_2^{n-1}), E_1^+\right)^h. \quad (4.3.54)$$

Similarly on choosing $\chi_2 = u_2^h$ in (4.2.3b), we also have

$$\frac{\gamma}{2}\left(|E_2^+|_1^2 + |U_2^n|_1^2 - |u_2^h|_1^2\right) \leq \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_2^n}{\partial t} - U_2^{n-1} + D((1-\beta)U_1^n + \beta U_1^{n-1}), E_2^+\right)^h. \quad (4.3.55)$$

Adding (4.3.54) to (4.3.53), when $i = 1$, gives

$$\begin{aligned}
&\frac{\gamma}{2}\left(|E_1|_1^2 + |E_1^+|_1^2 - |U_1|_1^2 + |u_1^h|_1^2 + |U_1^n|_1^2 - |u_1^h|_1^2\right) \\
&\quad \leq -\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_1^h}{\partial t}, E_1\right)^h + \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_1^n}{\partial t}, E_1^+\right)^h + (u_1^h, E_1)^h - (U_1^{n-1}, E_1^+)^h \\
&\qquad - D(u_2^h + 1, E_1)^h + D(\beta U_2^n + (1-\beta)U_2^{n-1}, E_1^+)^h, \qquad (4.3.56)
\end{aligned}$$

and adding (4.3.55) to (4.3.53), when $i = 2$, gives

$$\begin{aligned}
&\frac{\gamma}{2}\left(|E_2|_1^2 + |E_2^+|_1^2 - |U_2|_1^2 + |u_2^h|_1^2 + |U_2^n|_1^2 - |u_2^h|_1^2\right) \\
&\quad \leq -\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_2^h}{\partial t}, E_2\right)^h + \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_2^n}{\partial t}, E_2^+\right)^h + (u_2^h, E_2)^h - (U_2^{n-1}, E_2^+)^h \\
&\qquad - D(u_1^h + 1, E_2)^h + D((1-\beta)U_1^n + \beta U_1^{n-1}, E_2^+)^h. \qquad (4.3.57)
\end{aligned}$$

Noting (4.3.52), E_i^+ can be expressed as

$$E_i^+ = E_i + (U_i - U_i^n),$$

the first two terms in the right-hand side of (4.3.56) and (4.3.57) become

$$\begin{aligned} &-\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t}, E_i\right)^h + \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_i^n}{\partial t}, E_i^+\right)^h \\ &= -\left(\widehat{\mathcal{G}}_N^h \frac{\partial u_i^h}{\partial t} - \widehat{\mathcal{G}}_N^h \frac{\partial U_i^n}{\partial t}, E_i\right)^h + \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_i^n}{\partial t}, U_i - U_i^n\right)^h \\ &= -\left(\widehat{\mathcal{G}}_N^h \frac{\partial E_i}{\partial t}, E_i\right)^h + \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_i^n}{\partial t}, U_i - U_i^n\right)^h. \end{aligned} \tag{4.3.58}$$

The third and fourth terms can be rewritten as

$$\begin{aligned} (u_i^h, E_i)^h - (U_i^{n-1}, E_i^+)^h &= (u_i^h - U_i^{n-1}, E_i)^h - (U_i^{n-1}, U_i - U_i^n)^h \\ &= (E_i^-, E_i)^h - (U_i^{n-1}, U_i - U_i^n)^h. \end{aligned} \tag{4.3.59}$$

The fifth and the sixth terms of (4.3.56) and (4.3.57) can be expressed respectively as

$$\begin{aligned} &-D(u_2^h + 1, E_1)^h + D(\beta U_2^n + (1-\beta)U_2^{n-1}, E_1^+)^h \\ &= -D(\beta(u_2^h - U_2^n) + (1-\beta)(u_2^h - U_2^{n-1}) + 1, E_1)^h \\ &\quad + D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1 - U_1^n)^h \\ &= -D(\beta E_2^+ + (1-\beta)E_2^- + 1, E_1)^h + D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1 - U_1^n)^h, \end{aligned} \tag{4.3.60}$$

and

$$\begin{aligned} &-D(u_1^h + 1, E_2)^h + D((1-\beta)U_1^n + \beta U_1^{n-1}, E_2^+)^h \\ &= -D((1-\beta)(u_1^h - U_1^n) + \beta(u_1^h - U_1^{n-1}) + 1, E_2)^h \\ &\quad + D((1-\beta)U_1^n + \beta U_1^{n-1}, U_2 - U_2^n)^h \\ &= -D((1-\beta)E_1^+ + \beta E_1^- + 1, E_2)^h + D((1-\beta)U_1^n + \beta U_1^{n-1}, U_2 - U_2^n)^h. \end{aligned} \tag{4.3.61}$$

Noting that $(1, E_i)^h = 0$ for $i = 1, 2$, and (4.3.58)-(4.3.61), (4.3.56) becomes

$$\begin{aligned} &\frac{\gamma}{2}\left(|E_1|_1^2 + |E_1^+|_1^2\right) + \frac{1}{2}\frac{d}{dt}(\widehat{\mathcal{G}}_N^h E_1, E_1)^h \\ &\le (E_1^-, E_1)^h - D(\beta E_2^+ + (1-\beta)E_2^-, E_1)^h + \frac{\gamma}{2}\left(|U_1|_1^2 - |U_1^n|_1^2\right) \\ &\quad + \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_1^n}{\partial t} - U_1^{n-1} + D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1 - U_1^n\right)^h \end{aligned} \tag{4.3.62}$$

and (4.3.57) becomes

$$
\begin{aligned}
&\frac{\gamma}{2}\left(|E_2|_1^2 + |E_2^+|_1^2\right) + \frac{1}{2}\frac{d}{dt}(\widehat{\mathcal{G}}_N^h E_2, E_2)^h \\
&\quad \le\ (E_2^-, E_2)^h - D((1-\beta)E_1^+ + \beta E_1^-, E_2)^h + \frac{\gamma}{2}\left(|U_2|_1^2 - |U_2^n|_1^2\right) \\
&\qquad + \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_2^n}{\partial t} - U_2^{n-1} + D((1-\beta)U_1^n + \beta U_1^{n-1}, U_2 - U_2^n\right)^h .
\end{aligned} \tag{4.3.63}
$$

Adding (4.3.62) to (4.3.63) yields

$$
\begin{aligned}
&\frac{\gamma}{2}\left(|E_1|_1^2 + |E_2|_1^2 + |E_1^+|_1^2 + |E_2^+|_1^2\right) + \frac{1}{2}\left(\frac{d}{dt}(\widehat{\mathcal{G}}_N^h E_1, E_1)^h + \frac{d}{dt}(\widehat{\mathcal{G}}_N^h E_2, E_2)^h\right) \\
&\le\ (E_1^-, E_1)^h + (E_2^-, E_2)^h \\
&\quad - D(\beta E_2^+ + (1-\beta)E_2^-, E_1)^h - D((1-\beta)E_1^+ + \beta E_1^-, E_2)^h \\
&\quad + \frac{\gamma}{2}\left(|U_1|_1^2 + |U_2|_1^2\right) - \frac{\gamma}{2}\left(|U_1^n|_1^2 + |U_2^n|_1^2\right) \\
&\quad + \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_1^n}{\partial t}, U_1 - U_1^n\right)^h + \left(\widehat{\mathcal{G}}_N^h \frac{\partial U_2^n}{\partial t}, U_2 - U_2^n\right)^h \\
&\quad - (U_1^{n-1}, U_1 - U_1^{n-1})^h - (U_2^{n-1}, U_2 - U_2^n)^h \\
&\quad + D(\beta U_2^n + (1-\beta)U_2^{n-1}, U_1 - U_1^n)^h + D((1-\beta)U_1^n + \beta U_1^{n-1}, U_2 - U_2^n)^h .
\end{aligned} \tag{4.3.64}
$$

Rewriting (4.3.64), we thus obtain the desired result (4.3.51). □

Lemma 4.3.3 Let $a, b, c : (0, T) \to [0, +\infty]$ be measurable functions with a^2 being also absolutely continuous on $[0, T]$. If they satisfy the differential inequality

$$
\frac{d}{dt}a^2 + 2\lambda a^2 \ \le\ b^2 + 2ca, \qquad \text{for } a.e.\, t \in (0, T), \tag{4.3.65}
$$

depending on the parameter $\lambda \in \mathbb{R}$, then we have that

$$
\max_{t\in[0,T]} \mathrm{e}^{\lambda t} a(t) \ \le\ \left[(a(0))^2 + \int_0^T \mathrm{e}^{2\lambda t}(b(t))^2\, dt\right]^{\frac{1}{2}} + \int_0^T \mathrm{e}^{\lambda t} c(t)\, dt. \tag{4.3.66}
$$

Proof: See Lemma 3.7 in Nochetto, Savaré, and Verdi [42]. □

Theorem 4.3.4 Let the assumptions and notation of Lemma 4.3.2 hold. We have, for $n \to N$, $\alpha := \frac{1}{\gamma}$, that

$$\begin{aligned}\max_{t\in[0,t_n]} e^{-\frac{\alpha(1+D^2)}{2}t}\Big(\|E_1(\cdot,t)\|_{-h}^2 + \|E_2(\cdot,t)\|_{-h}^2\Big) &\\ \le \left[1+\left(8\alpha\left(\frac{3}{4}+D^2\right)^2 t_N\right)^{\frac{1}{2}}\right]^2 \sum_{n=1}^{N} e^{-\alpha(1+D^2)t_{n-1}}(\Delta t)^2\widehat{\mathcal{E}}^n &\\ \le C(T)(\Delta t)^2, & \qquad (4.3.67)\end{aligned}$$

$$\begin{aligned}\gamma\int_0^{t_N}\Big(|E_1^+(\cdot,t)|_1^2+|E_2^+(\cdot,t)|_1^2\Big)dt &\le 4\alpha(1+D^2)\int_0^{t_N}\Big(\|E_1(\cdot,t)\|_{-h}^2+\|E_1(\cdot,t)\|_{-h}^2\Big)\,dt\\ &\quad + 21\sum_{n=1}^{N}(\Delta t)^2\widehat{\mathcal{E}}^n\\ &\le C(T)(\Delta t)^2. \qquad (4.3.68)\end{aligned}$$

Proof: Noting (4.3.52), (4.3.3), (3.1.5) and the Cauchy-Schwarz inequality, for $i = 1,2$, $t \in (t_{n-1}, t_n)$, we obtain

$$\begin{aligned}(E_i^-, E_i)^h &= (u_i^h - U_i^{n-1}, E_i)^h\\ &= \left(\frac{1}{2}u_i^h - \frac{1}{2}U_i + \frac{1}{2}u_i^h - \frac{1}{2}U_i^n + \frac{1}{2}U_i + \frac{1}{2}U_i^n - U_i^{n-1}, E_i\right)^h\\ &= \left(\frac{1}{2}(E_i + E_i^+) + \frac{1}{2}(U_i + U_i^n) - U_i^{n-1}, E_i\right)^h\\ &= \frac{1}{2}(E_i + E_i^+, E)^h + \frac{1}{2}(U_i - U_i^{n-1} + U_i^n - U_i^{n-1}, E_i)^h\\ &= \frac{1}{2}(E_i + E_i^+, E)^h + \frac{1}{2}\left(\Delta t(1-\mu)\frac{\partial U_i}{\partial t} + \Delta t\frac{\partial U_i}{\partial t}, E_i\right)^h\\ &= \frac{1}{2}(E_i + E_i^+, E_i)^h + \frac{1}{2}((2-\mu)\Delta t\frac{\partial U_i}{\partial t}, E_i)^h\\ &= \frac{1}{2}\Big[(\nabla E_i, \nabla\widehat{\mathcal{G}}_N^h E_i) + (\nabla E_i^+, \nabla\widehat{\mathcal{G}}_N^h E_i)\Big] + \frac{1}{2}(2-\mu)\Delta t\left(\nabla\frac{\partial U_i}{\partial t}, \nabla\widehat{\mathcal{G}}_N^h E_i\right)\\ &\le \frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\Big(\|E_i\|_{-h}(|E_i|_1 + |E_i^+|_1) + \frac{1}{2}(2-\mu)\Delta t\left|\frac{\partial U_i}{\partial t}\right|_1\|E_i\|_{-h}\Big)].\end{aligned}$$

(4.3.69)

Noting the Young's inequality (2.1.8) and $(a+b)^2 \le 2(a^2+b^2)$ gives

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\|E_i\|_{-h}\Big(|E_i|_1 + |E_i^+|_1\Big) \le \frac{\alpha}{2}\|E_i\|_{-h}^2 + \frac{\gamma}{4}\Big(|E_i|_1^2 + |E_i^+|_1^2\Big), i = 1,2. \quad (4.3.70)$$

Noting (4.3.28) yields

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}(2-\mu)\Delta t\left|\frac{\partial U_i}{\partial t}\right|_1\|E_i\|_{-h} \le (2\alpha)^{\frac{1}{2}}\left(1-\frac{\mu}{2}\right)\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\|E_i\|_{-h}, i=1,2. \tag{4.3.71}$$

Substituting (4.3.70) and (4.3.71) into (4.3.69), we have, for $i=1,2$,

$$(E_i^-,E_i)^h \le \frac{\alpha}{2}\|E_i\|_{-h}^2+\frac{\gamma}{4}\Big(|E_i|_1^2+|E_i^+|_1^2\Big)+(2\alpha)^{\frac{1}{2}}\left(1-\frac{\mu}{2}\right)\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\|E_i\|_{-h}. \tag{4.3.72}$$

Adding (4.3.72) when $i=1$ to (4.3.72) when $i=2$ gives

$$\begin{aligned}(E_1^-,E_1)^h+(E_2^-,E_2)^h \le\ & \frac{\alpha}{2}\Big(\|E_1\|_{-h}^2+\|E_2\|_{-h}^2\Big)+\frac{\gamma}{4}\Big(|E_1|_1^2+|E_2|_1^2+|E_1^+|_1^2+|E_2^+|_1^2\Big)\\ &+(2\alpha)^{\frac{1}{2}}\left(1-\frac{\mu}{2}\right)\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}(\|E_1\|_{-h}+\|E_2\|_{-h}).\end{aligned} \tag{4.3.73}$$

In the same way as (4.3.69), for $i=1,2, i\neq j$, we also have that

$$(E_j^-,E_i)^h \le \frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\left(\|E_i\|_{-h}\left(|E_j|_1+|E_j^+|_1\right)+\frac{1}{2}(2-\mu)\Delta t\left|\frac{\partial U_j}{\partial t}\right|_1\|E_i\|_{-h}\right). \tag{4.3.74}$$

Noting the Young's inequality (2.1.8) and $(a+b)^2\le 2(a^2+b^2)$, for $i,j=1,2,\ i\neq j$ we obtain

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\|E_i\|_{-h}\Big(|E_j|_1+|E_j^+|_1\Big) \le \frac{\alpha D}{2}\|E_i\|_{-h}^2+\frac{\gamma}{4D}\Big(|E_j|_1^2+|E_j^+|_1^2\Big). \tag{4.3.75}$$

Noting (4.3.28), for $i,j=1,2,i\neq j$, it follows that

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}(2-\mu)\Delta t\left|\frac{\partial U_j}{\partial t}\right|_1\|E_i\|_{-h} \le (2\alpha)^{\frac{1}{2}}\left(1-\frac{\mu}{2}\right)\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\|E_i\|_{-h}. \tag{4.3.76}$$

Substituting (4.3.75) and (4.3.76) into (4.3.74) to obtain

$$\begin{aligned}|D(1-\beta)(E_2^-,E_1)^h| \le\ & \frac{\alpha D^2(1-\beta)}{2}\|E_1\|_{-h}^2+\frac{\gamma(1-\beta)}{4}\Big(|E_2|_1^2+|E_2^+|_1^2\Big)\\ &+D(1-\beta)(2\alpha)^{\frac{1}{2}}\left(1-\frac{\mu}{2}\right)\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\|E_1\|_{-h},\end{aligned} \tag{4.3.77}$$

and

$$\begin{aligned}|D\beta(E_1^-,E_2)^h| \le\ & \frac{\alpha D^2\beta}{2}\|E_2\|_{-h}^2+\frac{\gamma\beta}{4}\Big(|E_1|_1^2+|E_1^+|_1^2\Big)\\ &+D\beta(2\alpha)^{\frac{1}{2}}\left(1-\frac{\mu}{2}\right)\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\|E_2\|_{-h}.\end{aligned} \tag{4.3.78}$$

Again, noting (4.3.52), (3.1.3a), (4.3.3), the Cauchy-Schwarz inequality and (3.1.5), we then have

$$
\begin{aligned}
(E_j^+, E_i)^h &= \left(\frac{1}{2}u_j^h - \frac{1}{2}U_j + \frac{1}{2}u_j^h - \frac{1}{2}U_j^n + \frac{1}{2}U_j - \frac{1}{2}U_j^n, E_i\right)^h \\
&= \left(\frac{1}{2}E_j + \frac{1}{2}E_j^+ + \frac{1}{2}(U_j - U_j^n), E_i\right)^h \\
&= \frac{1}{2}(E_j + E_j^+, E_i)^h + \frac{1}{2}\mu\Delta t\left(\frac{\partial U}{\partial t}, E_i\right)^h \\
&= \frac{1}{2}\left[(\nabla E_j, \nabla \widehat{\mathcal{G}}_N^h E_i) + (\nabla E_j^+, \nabla \widehat{\mathcal{G}}_N^h E_i)\right] + \frac{1}{2}\mu\Delta t\left(\nabla \frac{\partial U_j}{\partial t}, \nabla \widehat{\mathcal{G}}_N^h E_i\right) \\
&\le \frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\left(\|E_i\|_{-h}|E_j|_1 + \|E_i\|_{-h}|E_j^+|_1 + \frac{1}{2}\mu\Delta t\left|\frac{\partial U_j}{\partial t}\right|_1 \|E_i\|_{-h}\right).
\end{aligned}
\tag{4.3.79}
$$

Noting (4.3.28) gives

$$
\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\mu\Delta t\left|\frac{\partial U_j}{\partial t}\right|_1 \|E_i\|_{-h} \le (2\alpha)^{\frac{1}{2}}\frac{\mu}{2}\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\|E_i\|_{-h}. \tag{4.3.80}
$$

Substituting (4.3.75) and (4.3.80) into (4.3.79), it follows that

$$
|D\beta(E_2^+, E_1)^h| \le \frac{\alpha D^2\beta}{2}\|E_1\|_{-h}^2 + \frac{\gamma\beta}{4}\left(|E_2|_1^2 + |E_2^+|_1^2\right) + D\beta(2\alpha)^{\frac{1}{2}}\frac{\mu}{2}\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\|E_1\|_{-h}, \tag{4.3.81}
$$

and

$$
\begin{aligned}
|D(1-\beta)(E_1^+, E_2)^h| \le{}& \frac{\alpha D^2(1-\beta)}{2}\|E_2\|_{-h}^2 + \frac{\gamma(1-\beta)}{4}\left(|E_1|_1^2 + |E_1^+|_1^2\right) \\
&+ D(1-\beta)(2\alpha)^{\frac{1}{2}}\frac{\mu}{2}\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\|E_2\|_{-h}.
\end{aligned}
\tag{4.3.82}
$$

Substituting (4.3.77), (4.3.78), (4.3.81) and (4.3.82) into (4.3.73), we obtain

$$
\begin{aligned}
&(E_1^- - D(\beta E_2^+ + (1-\beta)E_2^-, E_1)^h + (E_2^- - D((1-\beta)E_1^+ + \beta E_1^-, E_2)^h \\
&\le \frac{\alpha(1+D^2)}{2}\left(\|E_1\|_{-h}^2 + \|E_2\|_{-h}^2\right) + \frac{\gamma}{2}\left(|E_1|_1^2 + |E_2|_1^2 + |E_1^+|_1^2 + |E_2^+|_1^2\right) \\
&\quad + D(2\alpha)^{\frac{1}{2}}\frac{\mu}{2}\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\left(\|E_1\|_{-h} + \|E_2\|_{-h}\right) \\
&\quad + (1+D)(2\alpha)^{\frac{1}{2}}\left(1 - \frac{\mu}{2}\right)\left(\Delta t\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\left(\|E_1\|_{-h} + \|E_2\|_{-h}\right).
\end{aligned}
\tag{4.3.83}
$$

Combining (4.3.83), (4.3.51) and (4.3.29) yields

$$
\begin{aligned}
\frac{d}{dt}\left(\|E_1\|_{-h}^2 + \|E_2\|_{-h}^2\right) \le{}& \alpha(1+D^2)\left(\|E_1\|_{-h}^2 + \|E_2\|_{-h}^2\right) + 2\mu\Delta t\,\widehat{\mathcal{E}} \\
&+ 2(2\alpha)^{\frac{1}{2}}\left(1 + D - \frac{\mu}{2}\right)\left(\Delta t\,\widehat{\mathcal{E}}\right)^{\frac{1}{2}}\left(\|E_1\|_{-h} + \|E_2\|_{-h}\right).
\end{aligned}
$$

Noting Lemma 4.3.3 with $\lambda = -\frac{\alpha(1+D^2)}{2}$, $a^2 = \|E_i\|_{-h}^2$, $b^2 = 2\mu\Delta t\widehat{\mathcal{E}}$ and $c = (2\alpha)^{\frac{1}{2}}(1+D-\frac{\mu}{2})(\Delta t\widehat{\mathcal{E}})^{\frac{1}{2}}$ to have

$$\begin{aligned}
&\max_{t\in[0,t_m]} \mathrm{e}^{-\frac{\alpha(1+D^2)}{2}t}\Big(\|E_1(.,t)\|_{-h} + \|E_2(.,t)\|_{-h}\Big) \\
&\le \left(2\int_0^{t_N} \mathrm{e}^{-\alpha(1+D^2)t}\mu\Delta t\widehat{\mathcal{E}}(t)\,dt\right)^{\frac{1}{2}} + 2(2\alpha)^{\frac{1}{2}}\int_0^{t_m} \mathrm{e}^{-\frac{\alpha(1+D^2)}{2}t}\left(1+D-\frac{\mu}{2}\right)\left(\Delta t\widehat{\mathcal{E}}(t)\right)^{\frac{1}{2}} dt \\
&\le \left(\sum_{n=1}^{N} \mathrm{e}^{-\alpha(1+D^2)t_{n-1}}(\Delta t)^2\widehat{\mathcal{E}}^n\right)^{\frac{1}{2}} + \left(8\alpha\left(\frac{3}{4}+D\right)^2 t_N\sum_{n=1}^{N} \mathrm{e}^{-\alpha(1+D^2)t_{n-1}}(\Delta t)^2\widehat{\mathcal{E}}^n\right)^{\frac{1}{2}} \\
&\le \left[1+\left(8\alpha\left(\frac{3}{4}+D\right)^2 t_N\right)^{\frac{1}{2}}\right]\left(\sum_{n=1}^{N} \mathrm{e}^{-\alpha(1+D^2)t_{n-1}}(\Delta t)^2\widehat{\mathcal{E}}^n\right)^{\frac{1}{2}}. \qquad (4.3.84)
\end{aligned}$$

Then the desired result (4.3.67) follows on noting (4.3.84) and (4.3.30).

Now we use the same argument, but keep some of the "kick-back" term.

Reconsidering (4.3.69) and noting the Young's inequality (2.1.8), the first and the second term of the right-hand side of (4.3.69) can be expressed as, for $i = 1, 2$,

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\|E_i\|_{-h}|E_i|_1 \le \frac{\alpha}{4}\|E_i\|_{-h}^2 + \frac{\gamma}{4}|E_i|_1^2, \qquad (4.3.85)$$

and

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\|E_i\|_{-h}|E_i^+|_1 \le \frac{\alpha}{2}\|E_i\|_{-h}^2 + \frac{\gamma}{8}|E_i^+|_1^2. \qquad (4.3.86)$$

Noting (4.3.71) and the Young's inequality (2.1.8) to give

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}(2-\mu)\Delta t\left|\frac{\partial U_i}{\partial t}\right|\|E_i\|_{-h} \le \frac{\alpha}{4}\|E_i\|_{-h}^2 + 2\left(1-\frac{\mu}{2}\right)^2\Delta t\,\widehat{\mathcal{E}}. \qquad (4.3.87)$$

Combining (4.3.85) - (4.3.87) and (4.3.69), when $i = 1, 2$, we have that

$$\begin{aligned}
&(E_1^-, E_1)^h + (E_2^-, E_2)^h \\
&\le \alpha\Big(\|E_1\|_{-h}^2 + \|E_2\|_{-h}^2\Big) + \frac{\gamma}{8}\Big(2\left(|E_1|_1^2 + |E_2|_1^2\right) + |E_1^+|_1^2 + |E_2^+|_1^2\Big) + 4\left(1-\frac{\mu}{2}\right)^2\Delta t\,\widehat{\mathcal{E}}.
\end{aligned} \qquad (4.3.88)$$

Considering the right-hand side of (4.3.74). For $i = 1, 2, i \ne j$, we have on noting the Cauchy's inequality (2.1.9) and (4.3.76) that

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\|E_i\|_{-h}|E_j|_1 \le \frac{\alpha D}{4}\|E_i\|_{-h}^2 + \frac{\gamma}{4D}|E_j|_1^2, \qquad (4.3.89)$$

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\|E_i\|_{-h}|E_j^+|_1 \;\leq\; \frac{\alpha D}{2}\|E_i\|_{-h}^2 + \frac{\gamma}{8D}|E_j^+|_1^2, \tag{4.3.90}$$

and

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}(2-\mu)\left|\frac{\partial U_j}{\partial t}\right|_1 \|E_i\|_{-h} \;\leq\; \frac{\alpha D}{4}\|E_i\|_{-h}^2 + \frac{2}{D}\left(1-\frac{\mu}{2}\right)^2 \Delta t\,\widehat{\mathcal{E}}. \tag{4.3.91}$$

Combining (4.3.74) and (4.3.89) - (4.3.91), we then obtain

$$\begin{aligned}&|D(1-\beta)(E_2^-, E_1)^h|\\ &\leq\; \alpha D^2(1-\beta)\|E_1\|_{-h}^2 + \frac{\gamma(1-\beta)}{8}\left(2|E_2|_1^2 + |E_2^+|_1^2\right) + 2(1-\beta)\left(1-\frac{\mu}{2}\right)^2 \Delta t\,\widehat{\mathcal{E}},\end{aligned} \tag{4.3.92}$$

and

$$|D\beta(E_1^-, E_2)^h| \;\leq\; \alpha D^2\beta\|E_2\|_{-h}^2 + \frac{\gamma\beta}{8}\left(2|E_1|_1^2 + |E_1^+|_1^2\right) + 2\beta\left(1-\frac{\mu}{2}\right)^2 \Delta t\,\widehat{\mathcal{E}}. \tag{4.3.93}$$

Now consider the last term on the right-hand side of (4.3.79) yields on noting (4.3.80) and the Young's inequality (2.1.8) that

$$\frac{1}{2}(\gamma\alpha)^{\frac{1}{2}}\mu\Delta t\left|\frac{\partial U_j}{\partial t}\right|_1 \|E_i\|_{-h} \;\leq\; \frac{\alpha D}{4}\|E_i\|_{-h}^2 + \frac{2}{D}\left(\frac{\mu}{2}\right)^2 \Delta t\,\widehat{\mathcal{E}}. \tag{4.3.94}$$

Noting (4.3.79), (4.3.89), (4.3.90) and (4.3.94) to obtain

$$|D\beta(E_2^+, E_1)^h| \;\leq\; \alpha D^2\beta\|E_1\|_{-h}^2 + \frac{\gamma\beta}{8}\left(2|E_2|_1^2 + |E_2^+|_1^2\right) + 2\beta\left(\frac{\mu}{2}\right)^2 \Delta t\,\widehat{\mathcal{E}}, \tag{4.3.95}$$

and

$$\begin{aligned}&|D(1-\beta)(E_1^+, E_2)^h|\\ &\leq\; \alpha D^2(1-\beta)\|E_2\|_{-h}^2 + \frac{\gamma(1-\beta)}{8}\left(2|E_1|_1^2 + |E_1^+|_1^2\right) + 2(1-\beta)\left(\frac{\mu}{2}\right)^2 \Delta t\,\widehat{\mathcal{E}}.\end{aligned} \tag{4.3.96}$$

Combining (4.3.51), (4.3.29), (4.3.88), (4.3.92), (4.3.93), (4.3.95) and (4.3.96) gives

$$\begin{aligned}&\frac{\gamma}{4}\left(|E_1^+|_1^2 + |E_2^+|_1^2\right) + \frac{1}{2}\frac{d}{dt}\left(\|E_1\|_{-h}^2 + \|E_2\|_{-h}^2\right)\\ &\leq\; \alpha(1+D^2)\left(\|E_1\|_{-h}^2 + \|E_2\|_{-h}^2\right) + \left[6\left(1-\frac{\mu}{2}\right)^2 + \frac{\mu^2}{2} + \mu\right]\Delta t\,\widehat{\mathcal{E}}.\end{aligned} \tag{4.3.97}$$

Integrating (4.3.97) from 0 to t_N yields

$$\frac{\gamma}{4}\int_0^{t_N}\left(|E_1^+|_1^2+|E_2^+|_1^2\right)dt$$
$$\leq \alpha(1+D^2)\int_0^{t_N}\left(\|E_1\|_{-h}^2+\|E_2\|_{-h}^2\right)dt+\int_0^{t_N}\left[6\left(1-\frac{\mu}{2}\right)^2+\frac{\mu^2}{2}+\mu\right]\Delta t\,\widehat{\mathcal{E}}\,dt. \tag{4.3.98}$$

Considering the last term of the right-hand side of (4.3.98), noting (4.3.1), we obtain that

$$\begin{aligned}\int_0^{t_N}\left[6\left(1-\frac{\mu}{2}\right)^2+\frac{\mu^2}{2}+\mu\right]\Delta t\,\widehat{\mathcal{E}}\,dt &\leq \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\left[6\left(1-\frac{\mu}{2}\right)+\frac{\mu}{2}+\mu\right]\Delta t\,\widehat{\mathcal{E}}^n\,dt\\ &= \sum_{n=1}^{N}\left(\frac{9\,\Delta t}{2}+\frac{\Delta t}{4}+\frac{\Delta t}{2}\right)\Delta t\,\widehat{\mathcal{E}}^n\\ &= \frac{21}{4}\sum_{n=1}^{N}(\Delta t)^2\widehat{\mathcal{E}}^n,\end{aligned}$$

then (4.3.98) becomes

$$\frac{\gamma}{4}\int_0^{t_N}\left(|E_1^+|_1^2+|E_2^+|_1^2\right)dt \leq \alpha(1+D^2)\int_0^{t_N}\left(\|E_1\|_{-h}^2+\|E_2\|_{-h}^2\right)dt+\frac{21}{4}\sum_{n=1}^{N}(\Delta t)^2\widehat{\mathcal{E}}^n.$$

Therefore we have that

$$\gamma\int_0^{t_N}\left(|E_1^+|_1^2+|E_2^+|_1^2\right)dt \leq 4\alpha(1+D^2)\int_0^{t_N}\left(\|E_1\|_{-h}^2+\|E_2\|_{-h}^2\right)dt+21\sum_{n=1}^{N}(\Delta t)^2\widehat{\mathcal{E}}^n. \tag{4.3.99}$$

Then the desired result (4.3.68) follows on noting (4.3.99), (4.3.67) and (4.3.30). □

Theorem 4.3.5 Let the assumptions and notation of Lemma 4.3.2 hold. Let $d \leq 2$ then we have that

$$\begin{aligned}&\|E_1^+\|_{L^2(0,T;H^1(\Omega))}+\|E_2^+\|_{L^2(0,T;H^1(\Omega))}+\|E_1\|_{L^\infty(0,T;(H^1(\Omega))')}+\|E_2\|_{L^\infty(0,T;(H^1(\Omega))')}\\ &\quad\leq C(T)\sum_{n=1}^{N}(\Delta t)^2\widehat{\mathcal{E}}^n\\ &\quad\leq C(T)(\Delta t)^2.\end{aligned} \tag{4.3.100}$$

Proof: Noting (4.3.1) and rearranging (4.3.97), it follows that $0 \leq \mu \leq 1$ and

$$\frac{\gamma}{2}\left(|E_1^+|_1^2+|E_2^+|_1^2\right)+\frac{d}{dt}\left(\|E_1\|_{-h}^2+\|E_2\|_{-h}^2\right)$$
$$\leq \frac{2(1+D^2)}{\gamma}\left(\|E_1\|_{-h}^2+\|E_2\|_{-h}^2\right)+2(6-5\mu+2\mu^2)\Delta t\widehat{\mathcal{E}}. \tag{4.3.101}$$

Noting the Grönwall 's inequality, (4.3.101) becomes

$$
\begin{aligned}
&\frac{\gamma}{2}\int_0^{t_N}\left(|E_1^+|_1^2+|E_2^+|_1^2\right)dt+\|E_1(t_N)\|_{-h}^2+\|E_2(t_N)\|_{-h}^2\\
&\le e^{\frac{2(1+D^2)}{\gamma}t_N}\int_0^{t_N} e^{-\frac{2(1+D^2)}{\gamma}t}\left(6-\frac{3\mu}{2}\right)\Delta t\widehat{\mathcal{E}}(t)\,dt\\
&= e^{\frac{2(1+D^2)}{\gamma}t_N}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} e^{-\frac{2(1+D^2)}{\gamma}t}\left(6-\frac{3\mu}{2}\right)\Delta t\widehat{\mathcal{E}}(t)\,dt\\
&= e^{\frac{2(1+D^2)}{\gamma}t_N}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} e^{-\frac{2(1+D^2)}{\gamma}t}\left(6\Delta t-\frac{3\Delta t}{4}\right)\Delta t\widehat{\mathcal{E}}(s)\,dt\\
&\le \frac{21}{4}e^{\frac{2(1+D^2)}{\gamma}(t_n-t_{n-1})}\sum_{n=1}^{N}(\Delta t)^2\widehat{\mathcal{E}}^n\\
&\le C(T)\sum_{n=1}^{N}(\Delta t)^2\widehat{\mathcal{E}}^n.
\end{aligned}
$$

Then the second inequality in (4.3.100) follows from noting (4.3.30). □

Theorem 4.3.6 The unique solution $\{U_1^n, U_2^n\}$ to Problem $(\mathbf{P}_\beta^h)$ satisfies the error bounds

$$
\begin{aligned}
\|u_1-U_1^n\|_{L^2(0,T;H^1(\Omega))}^2&+\|u_2-U_2^n\|_{L^2(0,T;H^1(\Omega))}^2+\|u_1-U_1\|_{L^\infty(0,T;(H^1(\Omega))')}^2\\
&+\|u_2-U_2\|_{L^\infty(0,T;(H^1(\Omega))')}^2\le C(T)\Big[(\Delta t)^2+h^2\Big]. \qquad (4.3.102)
\end{aligned}
$$

Proof: Combining (3.3.1) and (4.3.100) gives

$$
\begin{aligned}
&\|u_1-u_1^h\|_{L^2(0,T;H^1(\Omega))}^2+\|u_2-u_2^h\|_{L^2(0,T;H^1(\Omega))}^2+\|E_1^+\|_{L^2(0,T;H^1(\Omega))}^2\\
&+\|E_2^+\|_{L^2(0,T;H^1(\Omega))}^2+\|u_1-u_1^h\|_{L^\infty(0,T;(H^1(\Omega))')}^2+\|u_2-u_2^h\|_{L^\infty(0,T;(H^1(\Omega))')}^2\\
&+\|E_1\|_{L^\infty(0,T;(H^1(\Omega))')}+\|E_2\|_{L^\infty(0,T;(H^1(\Omega))')}\\
&\le C(T)\Big[(\Delta t)^2+h^2\Big]. \qquad (4.3.103)
\end{aligned}
$$

Therefore (4.3.102) follows from (4.3.103) on noting (4.3.52). □

Chapter 5

Numerical Experiment

In Section 5.1 we introduce two practical algorithms to solve the decoupled ($\beta = 0$) and coupled ($\beta = \frac{1}{2}$) schemes which are used to solve Problem ($\mathbf{P}^h_\beta$). The convergence theory for the coupled scheme is proven. In Section 5.2 we discuss the linear stability of the solution for Problem (**P4**). We also performed some investigative computational results for one and two dimensions in Section 5.3.

5.1 Practical Algorithm

We define $K^h_{m_i}$ as in Section 3.1 and $S^h_{m_i}$ as in Section 4.2.

<u>**Case** $\beta = 0$</u>

Using (4.2.4) and setting $\beta = 0$ in (4.2.1b) and (4.2.2b), we have decoupled schemes solving for $\{U^k_1, U^k_2\}$ as

$$\gamma(\nabla U^k_1, \nabla\chi - \nabla U^k_1) - (U^{k-1}_1, \chi - U^k_1)^h + \left(\widehat{\mathcal{G}}^h_N\left(\frac{U^k_1 - U^{k-1}_1}{\Delta t}\right), \chi - U^k_1\right)^h$$
$$+ D(U^{k-1}_2, \chi - U^k_1)^h \geq (\lambda_1, \chi - U^k_1)^h \quad \forall \chi \in K^h_{m_1}, \quad (5.1.1)$$

and

$$\gamma(\nabla U^k_2, \nabla\chi - \nabla U^k_2) - (U^{k-1}_2, \chi - U^k_2)^h + \left(\widehat{\mathcal{G}}^h_N\left(\frac{U^k_2 - U^{k-1}_2}{\Delta t}\right), \chi - U^k_2\right)^h$$
$$+ D(U^k_1, \chi - U^k_2)^h \geq (\lambda_2, \chi - U^k_2)^h \quad \forall \chi \in K^h_{m_2}. \quad (5.1.2)$$

Let $\mathcal{B}_1 : S^h_{m_1} \to S^h$ be such that for all $\xi \in S^h$

$$(\mathcal{B}_1(\varrho), \xi)^h := \gamma(\nabla \varrho, \nabla \xi) - (U_1^{k-1}, \xi)^h + \left(\widehat{\mathcal{G}}^h_N \left(\frac{\varrho - U_1^{k-1}}{\Delta t}\right), \xi\right)^h + D(U_2^{k-1}, \xi)^h, \tag{5.1.3a}$$

and $\mathcal{B}_2 : S^h_{m_2} \to S^h$ be such that for all $\xi \in S^h$

$$(\mathcal{B}_2(\varrho), \xi)^h := \gamma(\nabla \varrho, \nabla \xi) - (U_2^{k-1}, \xi)^h + \left(\widehat{\mathcal{G}}^h_N \left(\frac{\varrho - U_2^{k-1}}{\Delta t}\right), \xi\right)^h + D(U_1^{k}, \xi)^h. \tag{5.1.3b}$$

Then $(\mathbf{P}^h_\beta)$ (4.2.1a-c) and (4.2.2a-c) can be rewritten as:

Problem $(\mathbf{P}^h_{\beta=0})$ For $k = 1 \to N$, find $\{U_1^k, U_2^k, \lambda_1, \lambda_2\} \in K^h_{m_1} \times K^h_{m_2} \times S^h_{m_1} \times S^h_{m_2}$ such that

$$(\mathcal{B}_1(U_1^k) - \lambda_1, \chi - U_1^k)^h \geq 0, \tag{5.1.4a}$$

$$(\mathcal{B}_2(U_2^k) - \lambda_2, \chi - U_2^k)^h \geq 0. \tag{5.1.4b}$$

Introducing the indicator function of the set $\mathcal{J} := \{y \in \mathbb{R} : |y| \leq 1\}$ defined by

$$I_{\mathcal{I}}(y) := \begin{cases} 0 & \text{if } y \in \mathcal{J} \\ +\infty & \text{if } y \notin \mathcal{J}, \end{cases}$$

the problem can be restated as find $\{U_1^k, U_2^k\} \in K^h_{m_1} \times K^h_{m_2}$ such that

$$\mathcal{B}_1(U_1^k)(x_m) + \mathcal{A}(U_1^k(x_m)) - \lambda_1^k \ni 0 \quad m = 1 \to N, \tag{5.1.5a}$$

$$\mathcal{B}_2(U_2^k)(x_m) + \mathcal{A}(U_2^k(x_m)) - \lambda_2^k \ni 0 \quad m = 1 \to N, \tag{5.1.5b}$$

where $\mathcal{A} := \partial I_{\mathcal{I}}$ is the subdifferential of $I_{\mathcal{I}}$:

$$\partial I_{\mathcal{I}}(x) = \{\eta \in \mathbb{R} : I_{\mathcal{I}}(x) - I_{\mathcal{I}}(y) \leq \eta^T(x - y) \ \forall\, x, y \in \mathbb{R}\}.$$

We seek the solution to (5.1.5a) and then the solution to (5.1.5b). For completeness we adapt and repeat the general splitting algorithm of Lions and Mercier [38] to solve the system (5.1.5a,b). Copetti and Elliott [22] and Barrett and Blowey [3] have applied this algorithm to solve a single Cahn-Hilliard equation with a logarithmic free energy and the Cahn-Hilliard system with non-smooth free energy, respectively.

Multiplying (5.1.5a) - (5.1.5b) by $\mu \in \mathbb{R}^+$, adding $U_1^n(x_m)$ and $U_2^n(x_m)$ respectively to both sides of (5.1.5a) and (5.1.5b), then rearranging yields, for $i = 1, 2$,

$$U_i^k(x_m) + \mu\mathcal{A}(U_i^k(x_m)) \ni R_i^k(x_m) \qquad m = 1 \to N,$$

where

$$R_i^k := U_i^k - \mu\Big[\mathcal{B}_i(U_i^k) - \lambda_i^k\Big].$$

For later use we define also

$$X_i^k := 2U_i^k - R_i^k \equiv U_i^k + \mu\Big[\mathcal{B}_i(U_i^k) - \lambda_i^k\Big].$$

Note that

$$(X_1^k - U_1^k, \chi)^h := \mu\left[\gamma(\nabla U_1^k, \nabla\chi) + \left(-U_1^{k-1} + D\,U_2^{k-1} + \widehat{\mathcal{G}}_N^h\left(\frac{U_1^k - U_1^{k-1}}{\Delta t}\right), \chi\right)^h\right]$$
$$\forall \chi \in S_{m_1}^h,$$

and

$$(X_2^k - U_2^k, \chi)^h := \mu\left[\gamma(\nabla U_2^k, \nabla\chi) + \left(-U_2^{k-1} + D\,U_1^k + \widehat{\mathcal{G}}_N^h\left(\frac{U_2^k - U_2^{k-1}}{\Delta t}\right), \chi\right)^h\right]$$
$$\forall \chi \in S_{m_2}^h.$$

For k fixed, a natural iteration to find $\{U_1^k, U_2^k\}$ satisfying (5.1.4a,b) is for $i = 1, 2$ and $j \geq 0$:

Given $U_i^{k,j} \in S_{m_i}^h$. Set

$$R_i^{k,j} := U_i^{k,j} - \mu\Big[\mathcal{B}_i(U_i^{k,j}) - \lambda_i^{k,j}\Big] \in S_{m_i}^h. \tag{5.1.7}$$

Find $U_i^{k,j+\frac{1}{2}} \in K^h$ such that

$$U_i^{k,j+\frac{1}{2}}(x_m) + \mu\mathcal{A}(U_i^{k,j+\frac{1}{2}}(x_m)) \ni R_i^{k,j}(x_m) \qquad m = 1 \to N. \tag{5.1.8}$$

Find $U_i^{k,j+1} \in S_{m_i}^h$ such that

$$U_i^{k,j+1}(x_m) + \mu\Big[\mathcal{B}_i(U_i^{k,j+1}(x_m)) - \lambda_i^{k,j+1}\Big] = X_i^{k,j+1}(x_m), \tag{5.1.9}$$

where

$$X_i^{k,j+1} := 2U_i^{k,j+\frac{1}{2}} - R_i^{k,j}. \tag{5.1.10}$$

Note that (5.1.8) is

$$\left(U_i^{k,j+\frac{1}{2}}(x_m) - R_i^{k,p}(x_m)\right)\left(r - U_i^{k,j+\frac{1}{2}}(x_m)\right) \geq 0, \qquad \forall |r| \leq 1, \quad m = 1 \to N.$$

Well-posedness and convergence of (5.1.7) - (5.1.10) are proved in Barrett and Blowey [3].

Case $\beta = \frac{1}{2}$

Setting $\beta = \frac{1}{2}$ in (4.2.1b) and (4.2.2b), we have the following coupled to solve for $\{U_1^k, U_2^k\}$

$$\gamma(\nabla U_1^k, \nabla\chi - \nabla U_1^k) - (U_1^{k-1}, \chi - U_1^k)^h + \frac{D}{2}(U_2^k + U_2^{k-1}, \chi - U_1^k)^h \geq (W_1^k, \chi - U_1^k)^h \quad \forall \chi \in K_{m_1}^h, \quad (5.1.11a)$$

and

$$\gamma(\nabla U_2^k, \nabla\chi - \nabla U_2^k) - (U_2^{k-1}, \chi - U_2^k)^h + \frac{D}{2}(U_1^k + U_1^{k-1}, \chi - U_2^k)^h \geq (W_2^k, \chi - U_2^k)^h \quad \forall \chi \in K_{m_2}^h. \quad (5.1.11b)$$

Then for k fixed, multiplying (5.1.11a,b) by $\mu > 0$, adding $(U_i^k, \chi - U_i^k)^h$ to both sides and rearranging, on noting (4.2.1a) and (4.2.2a), it follows that $\{U_1^k, U_2^k, W_1^k, W_2^k\} \in K_{m_1}^h \times K_{m_2}^h \times S_{m_1}^h \times S_{m_2}^h$ satisfies

$$\left(\frac{U_1^k - U_1^{k-1}}{\Delta t}, \chi\right)^h + (\nabla W_1^k, \nabla\chi) = 0 \quad \forall \chi \in S_{m_1}^h, \quad (5.1.12a)$$

$$(U_1^k, \chi - U_1^k)^h \geq (R_1^k, \chi - U_1^k)^h \quad \forall \chi \in K_{m_1}^h, \quad (5.1.12b)$$

and

$$\left(\frac{U_2^k - U_2^{k-1}}{\Delta t}, \chi\right)^h + (\nabla W_2^k, \nabla\chi) = 0 \quad \forall \chi \in S_{m_2}^h, \quad (5.1.13a)$$

$$(U_2^k, \chi - U_2^k)^h \geq (R_2^k, \chi - U_2^k)^h \quad \forall \chi \in K_{m_2}^h, \quad (5.1.13b)$$

where $R_1^k \in S_{m_1}^h$ and $R_2^k \in S_{m_2}^h$ are such that

$$(R_1^k - U_1^k, \chi)^h := -\mu\left[\gamma(\nabla U_1^k, \nabla\chi) + \left(-U_1^{k-1} + \frac{D}{2}(U_2^k + U_2^{k-1}) - W_1^k, \chi\right)^h\right] \quad \forall \chi \in S_{m_1}^h, \quad (5.1.14a)$$

and

$$(R_2^k - U_2^k, \chi)^h := -\mu \left[\gamma(\nabla U_2^k, \nabla \chi) + \left(-U_2^{k-1} + \frac{D}{2}(U_1^k + U_1^{k-1}) - W_2^k, \chi \right)^h \right]$$
$$\forall \chi \in S_{m_2}^h, \quad (5.1.14b)$$

respectively.

We introduce also $X_1^k \in S_{m_1}^h$ and $X_2^k \in S_{m_2}^h$ such that

$$(X_1^k - U_1^k, \chi)^h := \mu \left[\gamma(\nabla U_1^k, \nabla \chi) + \left(-U_1^{k-1} + \frac{D}{2}(U_2^k + U_2^{k-1}) - W_1^k, \chi \right)^h \right]$$
$$\forall \chi \in S_{m_1}^h, \quad (5.1.15a)$$

and

$$(X_2^k - U_2^k, \chi)^h := \mu \left[\gamma(\nabla U_2^k, \nabla \chi) + \left(-U_2^{k-1} + \frac{D}{2}(U_1^k + U_1^{k-1}) - W_2^k, \chi \right)^h \right]$$
$$\forall \chi \in S_{m_2}^h. \quad (5.1.15b)$$

Note that $X_1^k = 2\,U_1^k - R_1^k$ and $X_2^k = 2\,U_2^k - R_2^k$. We use the above as a basis for constructing our iterative procedure:

For $k \geq 1$ set $U_1^{k,0} \equiv U_1^{k-1} \in K_{m_1}^h$ and $U_2^{k,0} \equiv U_2^{k-1} \in K_{m_2}^h$.

For $j \geq 0$ we define $R_1^{k,j} \in S_{m_1}^h$ and $R_2^{k,j} \in S_{m_2}^h$ such that

$$(R_1^{k,j} - U_1^{k,j}, \chi)^h = -\mu \left[\gamma(\nabla U_1^{k,j}, \nabla \chi) + \left(U_1^{k-1} + \frac{D}{2}(U_2^{k,j} + U_2^{k-1}) - W_1^{k,j}, \chi \right)^h \right]$$
$$\forall \chi \in S_{m_1}^h \quad (5.1.16a)$$

and

$$(R_2^{k,j} - U_2^{k,j}, \chi)^h = -\mu \left[\gamma(\nabla U_2^{k,j}, \nabla \chi) + \left(U_2^{k-1} + \frac{D}{2}(U_1^{k,j} + U_1^{k-1}) - W_2^{k,j}, \chi \right)^h \right]$$
$$\forall \chi \in S_{m_2}^h. \quad (5.1.16b)$$

Then we find $\left\{ U_1^{k,j+\frac{1}{2}}, U_2^{k,j+\frac{1}{2}} \right\} \in K^h \times K^h$ such that

$$\left(U_1^{k,j+\frac{1}{2}}(x_m) - R_1^{k,j}(x_m) \right) \left(r - U_1^{k,j+\frac{1}{2}}(x_m) \right) \geq 0, \qquad \forall |r| \leq 1, \quad m = 1 \to N,$$
$$(5.1.17a)$$

and

$$\left(U_2^{k,j+\frac{1}{2}}(x_m) - R_2^{k,j}(x_m)\right)\left(r - U_2^{k,j+\frac{1}{2}}(x_m)\right) \geq 0, \quad \forall |r| \leq 1, \quad m = 1 \to N, \tag{5.1.17b}$$

and find $\left\{U_1^{k,j+1}, U_1^{k,j+1}, W_1^{k,j+1}, W_2^{k,j+1}\right\} \in S_{m_1}^h \times S_{m_2}^h \times S^h \times S^h$ such that

$$\left(\frac{U_1^{k,j+1} - U_1^{k-1}}{\Delta t}, \chi\right)^h + (\nabla W_1^{k,j+1}, \nabla \chi) = 0 \quad \forall \chi \in S_{m_1}^h, \tag{5.1.18a}$$

$$\begin{aligned}(U_1^{k,j+1}, \chi)^h + \mu \left[\gamma(\nabla U_1^{k,j+1}, \nabla \chi) + \left(\frac{D}{2} U_2^{k,j+1} - W_1^{k,j+1}, \chi\right)^h\right] \\ = \left(X_1^{k,j+1} + \mu\left(U_1^{k-1} - \frac{D}{2}U_2^{k-1}\right), \chi\right)^h \quad \forall \chi \in K_{m_1}^h,\end{aligned} \tag{5.1.18b}$$

and

$$\left(\frac{U_2^{k,j+1} - U_2^{k-1}}{\Delta t}, \chi\right)^h + (\nabla W_2^{k,j+1}, \nabla \chi) = 0 \quad \forall \chi \in S_{m_2}^h, \tag{5.1.19a}$$

$$\begin{aligned}(U_2^{k,j+1}, \chi)^h + \mu \left[\gamma(\nabla U_2^{k,j+1}, \nabla \chi) + \left(\frac{D}{2} U_1^{k,j+1} - W_1^{k,j+1}, \chi\right)^h\right] \\ = \left(X_2^{k,j+1} - \mu\left(U_2^{k-1} - \frac{D}{2}U_1^{k-1}\right), \chi\right)^h \quad \forall \chi \in K_{m_2}^h;\end{aligned} \tag{5.1.19b}$$

where $X_1^{k,j+1} := 2\,U_1^{k,j+\frac{1}{2}} - R_1^{k,j}$ and $X_2^{k,j+1} := 2\,U_2^{k,j+\frac{1}{2}} - R_2^{k,j}$.

We now proved well-posedness of (5.1.16a) - (5.1.19b).

For each m existence of $\left\{U_1^{k,j+\frac{1}{2}}(x_m), U_2^{k,j+\frac{1}{2}}(x_m)\right\}$ in the variational inequality (5.1.17a,b) follows from the minimization problem

$$\mathcal{I}_1(r) = \min_{|r|\leq 1} \frac{1}{2}|r - R_1^{k,j}(x_m)|^2, \tag{5.1.20a}$$

and

$$\mathcal{I}_2(r) = \min_{|r|\leq 1} \frac{1}{2}|r - R_2^{k,j}(x_m)|^2. \tag{5.1.20b}$$

Assume that $U_1^{k,j+\frac{1}{2}}(x_m)$ is the solution of (5.1.20a). Let $r = U_1^{k,j+\frac{1}{2}} + \lambda(r - U_1^{k,j+\frac{1}{2}})$ and $0 < \lambda < 1$, then

$$\begin{aligned} 0 &\le \mathcal{I}(U_1^{k,j+\frac{1}{2}} + \lambda(r - U_1^{k,j+\frac{1}{2}})) - \mathcal{I}(U_1^{k,j+\frac{1}{2}}) \\ &= \frac{1}{2}|U_1^{k,j+\frac{1}{2}} + \lambda(r - U_1^{k,j+\frac{1}{2}}) - R_1^{k,j}|^2 - \frac{1}{2}|U_1^{k,j+\frac{1}{2}} - R_1^{k,j}|^2 \\ &= \frac{1}{2}\Big[\mathcal{I}_1(U_1^{k,j+\frac{1}{2}}) + 2\lambda(U_1^{k,j+\frac{1}{2}} - R_1^{k,j})(r - U_1^{k,j+\frac{1}{2}}) + \lambda^2(r - U_1^{k,j+\frac{1}{2}})^2 - \mathcal{I}_1(U_1^{k,j+\frac{1}{2}})\Big]. \end{aligned} \tag{5.1.21}$$

Dividing both sides of (5.1.21) by λ and then letting $\lambda \to 0^+$, it follows that

$$\left(U_1^{k,j+\frac{1}{2}} - R_1^{k,p}\right)\left(r - U_1^{k,j+\frac{1}{2}}\right) \ \ge\ 0.$$

In the same way, assuming that $U_2^{k,j+\frac{1}{2}}$ is the solution of (5.1.20b) and let $r = U_2^{k,j+\frac{1}{2}} + \lambda(r - U_2^{k,j+\frac{1}{2}})$, $0 < \lambda < 1$, we obtain (5.1.17b).

To prove uniqueness, let $\left\{U_1^{k,j+\frac{1}{2}}(x_m), U_2^{k,j+\frac{1}{2}}(x_m)\right\}$ and $\left\{U_{1*}^{k,j+\frac{1}{2}}(x_m), U_{2*}^{k,j+\frac{1}{2}}(x_m)\right\}$ be two different solutions of (5.1.17a,b) . Substituting $r = \left\{U_{1*}^{k,j+\frac{1}{2}}(x_m), U_{2*}^{k,j+\frac{1}{2}}(x_m)\right\}$ when $\left\{U_1^{k,j+\frac{1}{2}}(x_m), U_2^{k,j+\frac{1}{2}}(x_m)\right\}$ is the solution, and vice-versa, we have that

$$\left(U_1^{k,j+\frac{1}{2}}(x_m) - R_1^{k,p}(x_m)\right)\left(U_{1*}^{k,j+\frac{1}{2}}(x_m) - U_1^{k,j+\frac{1}{2}}(x_m)\right) \ \ge\ 0, \tag{5.1.22a}$$

$$\left(U_2^{k,j+\frac{1}{2}}(x_m) - R_2^{k,p}(x_m)\right)\left(U_{2*}^{k,j+\frac{1}{2}}(x_m) - U_2^{k,j+\frac{1}{2}}(x_m)\right) \ \ge\ 0, \tag{5.1.22b}$$

and

$$\left(U_{1*}^{k,j+\frac{1}{2}}(x_m) - R_1^{k,p}(x_m)\right)\left(U_1^{k,j+\frac{1}{2}}(x_m) - U_{1*}^{k,j+\frac{1}{2}}(x_m)\right) \ \ge\ 0, \tag{5.1.23a}$$

$$\left(U_{2*}^{k,j+\frac{1}{2}}(x_m) - R_2^{k,p}(x_m)\right)\left(U_2^{k,j+\frac{1}{2}}(x_m) - U_{2*}^{k,j+\frac{1}{2}}(x_m)\right) \ \ge\ 0. \tag{5.1.23b}$$

Adding (5.1.22a) and (5.1.23a), it follows that

$$-\left(U_1^{k,j+\frac{1}{2}}(x_m) - U_{1*}^{k,j+\frac{1}{2}}(x_m)\right)^2 \ \ge\ 0.$$

In the same way, adding (5.1.22b) and (5.1.23b), we also have

$$-\left(U_2^{k,j+\frac{1}{2}}(x_m) - U_{2*}^{k,j+\frac{1}{2}}(x_m)\right)^2 \ \ge\ 0.$$

Thus $\left\{U_1^{k,j+\frac{1}{2}}(x_m), U_2^{k,j+\frac{1}{2}}(x_m)\right\} = \left\{U_{1*}^{k,j+\frac{1}{2}}(x_m), U_{2*}^{k,j+\frac{1}{2}}(x_m)\right\}$ which is a contradiction.

It remains to show that there exists a unique solution to (5.1.18a,b) and (5.1.19a,b). It follows from (5.1.18a,b) that

$$W_1^{k,j+1} = -\widehat{\mathcal{G}}_N^h\left(\frac{U_1^{k,j+1}-U_1^{k-1}}{\Delta t}\right)+\lambda_1^{k,j+1}, \tag{5.1.24a}$$

$$W_2^{k,j+1} = -\widehat{\mathcal{G}}_N^h\left(\frac{U_2^{k,j+1}-U_2^{k-1}}{\Delta t}\right)+\lambda_2^{k,j+1}, \tag{5.1.24b}$$

where

$$\lambda_1^{k,j+1} = \fint \left[\mu^{-1}(U_1^{k,j+1}-X_1^{k,j+1})-U_1^{k,j+1}+\frac{D}{2}(U_2^{k,j+1}+U_2^{k-1})\right],$$

$$\lambda_2^{k,j+1} = \fint \left[\mu^{-1}(U_2^{k,j+1}-X_2^{k,j+1})-U_2^{k,j+1}+\frac{D}{2}(U_1^{k,j+1}+U_1^{k-1})\right].$$

Therefore (5.1.18a,b) and (5.1.19a,b) may be written equivalently as:

Find $\left\{U_1^{k,j+1},U_2^{k,j+1}\right\}\in S_{m_1}^h\times S_{m_2}^h$ such that

$$\begin{aligned}&\left(U_1^{k,j+1},\left(1-\fint\right)\chi\right)^h\\&+\mu\left[\gamma(\nabla U_1^{k,j+1},\nabla\chi)+\left(\widehat{\mathcal{G}}_N^h\left(\frac{U_1^{k,j+1}-U_1^{k-1}}{\Delta t}\right)+\frac{D}{2}U_2^{k,j+1},\left(1-\fint\right)\chi\right)^h\right]\\&\qquad=\left(B_1^{k,j+1},\left(1-\fint\right)\chi\right)^h\quad\forall\chi\in S^h,\end{aligned} \tag{5.1.25a}$$

and

$$\begin{aligned}&\left(U_2^{k,j+1},\left(1-\fint\right)\chi\right)^h\\&+\mu\left[\gamma(\nabla U_2^{k,j+1},\nabla\chi)+\left(\widehat{\mathcal{G}}_N^h\left(\frac{U_2^{k,j+1}-U_2^{k-1}}{\Delta t}\right)+\frac{D}{2}U_1^{k,j+1},\left(1-\fint\right)\chi\right)^h\right]\\&\qquad=\left(B_2^{k,j+1},\left(1-\fint\right)\chi\right)^h\quad\forall\chi\in S^h,\end{aligned} \tag{5.1.25b}$$

where

$$B_1^{k,j+1} := X_1^{k,j+1}+\mu\left(U_1^{k-1}-\frac{D}{2}U_1^{k-1}\right),$$

and

$$B_2^{k,j+1} := X_2^{k,j+1}+\mu\left(U_2^{k-1}-\frac{D}{2}U_2^{k-1}\right).$$

Existence of $\left\{U_1^{k,j+1}, U_2^{k,j+1}\right\} \in S_{m_1}^h \times S_{m_2}^h$ satisfying (5.1.25a,b) follows from the Euler-Lagrange equation of the minimization problem

$$
\begin{aligned}
\mathcal{I}(\chi_1,\chi_2) = \min_{\chi_i\in S_{m_i}^h, i=1,2} \Big\{ & |\chi_1|_h^2 + |\chi_2|_h^2 \\
& + \mu\Big[\gamma\Big(|\chi_1|_1^2 + |\chi_2|_1^2\Big) \\
& + \frac{1}{\Delta t}\Big(|\nabla\widehat{\mathcal{G}}_N^h(\chi_1 - U_1^{k-1})|_0^2 + |\nabla\widehat{\mathcal{G}}_N^h(\chi_2 - U_2^{k-1})|_0^2\Big) \\
& + D(\chi_1,\chi_2)^h\Big] \\
& - 2\Big[(B_1^{k,j+1},\chi_1)^h + (B_2^{k,j+1},\chi_2)^h\Big]\Big\}. \qquad (5.1.27)
\end{aligned}
$$

Assume that $\left\{U_1^{k,j+1}, U_2^{k,j+1}\right\} \in S_{m_1}^h \times S_{m_2}^h$ is the solution of (5.1.27). Let $S_0^h := \{\eta \in S^h : (\eta,1)^h = 0\}$, $\{\xi_1,\xi_2\} \in S_{m_1}^h \times S_{m_2}^h$, $\xi_1 = U_1^{k,j+1} + \lambda\,\chi_1$ where $\chi_1 \in S_0^h$, $\xi_2 = U_2^{k,j+1}$, and $|\lambda| > 0$ then

$$
\begin{aligned}
0 \le\ & \mathcal{I}(U_1^{k,j+1} + \lambda\,\chi_1, U_2^{k,j+1}) - \mathcal{I}(U_1^{k,j+1}, U_2^{k,j+1}) \\
=\ & |U_1^{k,j+1} + \lambda\,\chi_1|_h^2 - |U_1^{k,j+1}|_h^2 \\
& + \mu\Big[\gamma\Big(|U_1^{k,j+1} + \lambda\,\chi_1|_1^2 - |U_1^{k,j+1}|_1^2\Big) \\
& + \frac{1}{\Delta t}\Big(|\nabla\widehat{\mathcal{G}}_N^h(U_1^{k,j+1} + \lambda\,\chi_1 - U_1^{k-1})|_0^2 - |\nabla\widehat{\mathcal{G}}_N^h(U_1^{k,j+1} - U_1^{k-1})|_0^2\Big) \\
& - \Big(|U_1^{k,j+1} + \lambda\,\chi_1|_h^2 - |U_1^{k,j+1}|_h^2\Big) \\
& + \frac{D}{2}(U_1^{k,j+1} + \lambda\,\chi_1, U_2^{k,j+1})^h - \frac{D}{2}(U_1^{k,j+1}, U_2^{k,j+1})^h\Big] \\
& - 2\Big[(B_1^{k,j+1}, U_1^{k,j+1} + \lambda\,\chi_1)^h - (B_1^{k,j+1}, U_1^{k,j+1})^h\Big]. \qquad (5.1.28)
\end{aligned}
$$

In the same way as (4.2.31) - (4.2.34), we have

$$|U_1^{k,j+1} + \lambda\,\chi_1|_h^2 - |U_1^{k,j+1}|_h^2 = 2\lambda(U_1^{k,j+1},\chi_1)^h - \lambda^2|\chi_1|_h^2, \qquad (5.1.29)$$

$$(|U_1^{k,j+1} + \lambda(\chi_1 - U_1^{k,j+1})|_1^2 - |U_1^{k,j+1}|_1^2) = 2\lambda(\nabla U_1^{k,j+1},\nabla\chi_1) + \lambda^2|\chi_1|_1^2, \qquad (5.1.30)$$

$$
\begin{aligned}
& |\nabla\widehat{\mathcal{G}}_N^h(U_1^{k,j+1} + \lambda\,\chi_1 - U_1^{k-1})|_0^2 - |\nabla\widehat{\mathcal{G}}_N^h(U_1^{k,j+1} - U_1^{k-1})|_0^2 \\
& \qquad = 2\lambda(\widehat{\mathcal{G}}_N^h(U_1^{k,j+1} - U_1^{k-1}),\chi_1)^h + \lambda^2|\nabla\widehat{\mathcal{G}}_N^h\,\chi_1|_0^2, \qquad (5.1.31)
\end{aligned}
$$

$$D(U_1^{k,j+1} + \lambda\,\chi_1, U_2^{k,j+1})^h - D(U_1^{k,j+1}, U_2^{k,j+1})^h = \lambda\,D(U_2^{k,j+1},\chi_1), \qquad (5.1.32)$$

and

$$(B_1^{k,j+1}, U_1^{k,j+1} + \lambda\chi_1)^h - (B_1^{k,j+1}, U_1^{k,j+1})^h = \lambda(B_1^{k,j+1}, \chi_1). \qquad (5.1.33)$$

Substitute (5.1.29) - (5.1.33) into (5.1.28), then dividing both sides by 2λ, and finally letting $\lambda \to 0$, it follows that

$$(U_1^{k,j+1}, \chi_1)^h + \mu\left[\gamma(\nabla U_1^{k,j+1}, \nabla\chi_1) + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_1^{k,j+1} - U_1^{k-1}}{\Delta t}\right) + \frac{D}{2}U_2^{k,j+1}, \chi_1\right)^h\right] \qquad (5.1.34)$$

$$= (B_1^{k,j+1}, \chi_1)^h. \qquad (5.1.35)$$

Next, let $\chi \in S^h$ be arbitrary, and set $\chi_1 = \chi - \frac{1}{|\Omega|}(\chi_1, 1)$ so that $\chi_1 \in S_0^h$ hence from (5.1.34) we arrive at (5.1.25a).

In the same way as (5.1.28) - (5.1.34), assume that $\left\{U_1^{k,j+1}, U_2^{k,j+1}\right\} \in S_{m_1}^h \times S_{m_2}^h$ is the solution of (5.1.27). Let $\{\xi_1, \xi_2\} \in S_{m_1}^h \times S_{m_2}^h$, $\xi_1 = U_1^{k,j+1}$, $\xi_2 = U_2^{k,j+1} + \lambda\chi_2$ where $\chi_2 \in S_0^h$, and $|\lambda| > 0$. Then set $\chi_2 = \chi - \frac{1}{|\Omega|}(\chi, 1)$ so that $\chi_2 \in S_0^h$ hence we obtain (5.1.25b).

In order to prove uniqueness, let $\left\{U_1^{k,j+1}, U_2^{k,j+1}\right\}$ and $\left\{U_{1*}^{k,j+1}, U_{2*}^{k,j+1}\right\}$ be two solutions, define $\theta_1^U = U_1^{k,j+1} - U_{1*}^{k,j+1}$ and $\theta_2^U = U_2^{k,j+1} - U_{2*}^{k,j+1}$. Substituting $\chi = \theta_1^U$ into (5.1.25a) and $\chi = \theta_2^U$ into (5.1.25b), when $\left\{U_1^{k,j+1}, U_2^{k,j+1}\right\}$ is the solution, gives

$$\begin{aligned}&(U_1^{k,j+1}, \theta_1^U)^h\\ &\quad + \mu\left[\gamma(\nabla U_1^{k,j+1}, \nabla\theta_1^U) + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_1^{k,j+1} - U_1^{k-1}}{\Delta t}\right) + \frac{D}{2}U_2^{k,j+1} - \lambda_1, \theta_1^U\right)^h\right]\\ &\qquad = (B_1^{k,j+1}, \theta_1^U)^h, \end{aligned} \qquad (5.1.36a)$$

and

$$\begin{aligned}&(U_2^{k,j+1}, \theta_2^U)^h\\ &\quad + \mu\left[\gamma(\nabla U_2^{k,j+1}, \nabla\theta_2^U) + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_2^{k,j+1} - U_2^{k-1}}{\Delta t}\right) + \frac{D}{2}U_1^{k,j+1} - \lambda_2, \theta_2^U\right)^h\right]\\ &\qquad = (B_2^{k,j+1}, \theta_2^U)^h. \end{aligned} \qquad (5.1.36b)$$

Substituting $\chi = -\theta_1^U$ into (5.1.25a) and $\chi = -\theta_2^U$ into (5.1.25b), when $\left\{U_{1*}^{k,j+1}, U_{2*}^{k,j+1}\right\}$ is the solution, gives

$$
\begin{aligned}
&(U_{1*}^{k,j+1}, -\theta_1^U)^h \\
&\quad + \mu\left[\gamma(\nabla U_{1*}^{k,j+1}, -\nabla\theta_1^U) + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_{1*}^{k,j+1} - U_1^{k-1}}{\Delta t}\right) + \frac{D}{2}U_{2*}^{k,j+1} - \lambda_1, -\theta_1^U\right)^h\right] \\
&\qquad = (B_1^{k,j+1}, -\theta_1^U)^h, \quad (5.1.37a)
\end{aligned}
$$

and

$$
\begin{aligned}
&(U_{2*}^{k,j+1}, -\theta_2^U)^h \\
&\quad + \mu\left[\gamma(\nabla U_{2*}^{k,j+1}, -\nabla\theta_1^U) + \left(\widehat{\mathcal{G}}_N^h\left(\frac{U_{2*}^{k,j+1} - U_2^{k-1}}{\Delta t}\right) + \frac{D}{2}U_{1*}^{k,j+1} - \lambda_2, -\theta_2^U\right)^h\right] \\
&\qquad = (B_2^{k,j+1}, -\theta_2^U)^h. \quad (5.1.37b)
\end{aligned}
$$

Adding (5.1.36a) and (5.1.37a), it follows that

$$|\theta_1^U|_h^2 + \mu\left[\gamma|\theta_1^U|_1^2 + \frac{1}{\Delta t}\|\theta_1^U\|_{-h}^2 + \frac{D}{2}(\theta_2^U, \theta_1^U)^h\right] = 0. \qquad (5.1.38)$$

By (3.1.3a) and the inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$; $\epsilon = \frac{1}{\delta} > 0$

$$\left|\frac{D}{2}(\theta_2^U, \theta_1^U)^h\right| = D\left(\nabla\widehat{\mathcal{G}}_N^h\theta_1^U, \nabla\theta_2^U\right) \leq \frac{1}{\delta}\|\theta_1^U\|_{-h}^2 + \frac{\delta D^2}{16}|\theta_2^U|_1^2. \qquad (5.1.39)$$

Hence (5.1.38) becomes

$$|\theta_1^U|_h^2 + \mu\left[\gamma|\theta_1^U|_1^2 + \left(\frac{1}{\Delta t} - \frac{1}{\delta}\right)\|\theta_1^U\|_{-h}^2 - \frac{\delta D^2}{16}|\theta_2^U|_1^2\right] \leq 0. \qquad (5.1.40)$$

In the same way, by adding (5.1.36b) and (5.1.37b), we then have

$$|\theta_2^U|_h^2 + \mu\left[\gamma|\theta_2^U|_1^2 + \left(\frac{1}{\Delta t} - \frac{1}{\delta}\right)\|\theta_2^U\|_{-h}^2 - \frac{\delta D^2}{16}|\theta_1^U|_1^2\right] \leq 0. \qquad (5.1.41)$$

Now adding (5.1.40) and (5.1.41) to obtain

$$
\begin{aligned}
&|\theta_1^U|_h^2 + |\theta_2^U|_h^2 \\
&\quad + \mu\left[\left(\gamma - \frac{\delta D^2}{16}\right)\left(|\theta_1^U|_1^2 + |\theta_2^U|_1^2\right) + \left(\frac{1}{\Delta t} - \frac{1}{\delta}\right)\left(\|\theta_1^U\|_{-h}^2 + \|\theta_2^U\|_{-h}^2\right)\right] \leq 0.
\end{aligned}
$$

(5.1.42)

Taking $\delta = \Delta t$, then (5.1.42) becomes

$$|\theta_1^U|_h^2 + |\theta_2^U|_h^2 + \mu\left(\gamma - \frac{D^2\Delta t}{16}\right)\left(|\theta_1^U|_1^2 + |\theta_2^U|_1^2\right) \le 0.$$

Thus the uniqueness of $\left\{U_1^{k,j+1}, U_2^{k,j+1}\right\}$ follows for $\Delta t < \frac{16\gamma}{D^2}$ which holds from Theorem 4.2.1.

Finally, existence and uniqueness of $\left\{W_1^{k,j+1}, W_2^{k,j+1}\right\}$ follow from (5.1.24a,b). Hence the iterative procedure (5.1.16a) - (5.1.19b) is well-defined. Moreover, at each iteration one needs to solve only (i) a fixed linear system with constant coefficients and (ii) a decoupled nonlinear equation for each component at each mesh point.

We discuss the solution of (i); that is, (5.1.25a,b). Expanding $U_i, B_i, i = 1, 2$ in terms of the standard nodal basis function φ_i of the finite element space S^h yields

$$U_1^{k,j} = \sum_{m=1}^{N} \mathcal{U}_{1,m}^{k,j}\,\varphi_m(x), \qquad B_1^{k,j} = \sum_{m=1}^{N} \mathcal{B}_{1,m}^{k,j}\,\varphi_m(x), \tag{5.1.43a}$$

$$U_2^{k,j} = \sum_{m=1}^{N} \mathcal{U}_{2,m}^{k,j}\,\varphi_m(x), \qquad B_2^{k,j} = \sum_{m=1}^{N} \mathcal{B}_{2,m}^{k,j}\,\varphi_m(x). \tag{5.1.43b}$$

We introduce the $N \times N$ stiffness matrix $\mathcal{K} \equiv \{\mathcal{K}_{m,m'}\}$ and mass matrix $\mathcal{M} \equiv \{\mathcal{M}_{m,m'}\}$, where

$$\mathcal{K}_{m,m'} := (\nabla\varphi_m, \nabla\varphi_{m'}), \qquad \mathcal{M}_{m,m'} := (\varphi_m, \varphi_{m'})^h = \omega_m\,\delta_{mm'},$$

and the $N \times 1$ vector $\underline{e} \equiv 1_m \in \mathbb{R}^N$ defined by

$$\underline{e}_m = (\varphi_m, 1)^h.$$

Observe the algebraic form of (3.1.3a): Given $\underline{v} \in \mathbb{R}^N$ such that

$$\underline{e}^T\underline{v} = 0,$$

find $\hat{\underline{v}} \in \mathbb{R}^N$ satisfying

$$\mathcal{K}\hat{\underline{v}} = \mathcal{M}\underline{v}, \tag{5.1.44a}$$

$$\text{and } \underline{e}^T\hat{\underline{v}} = 0. \tag{5.1.44b}$$

There exists a unique solution $\hat{\underline{v}}$ to (5.1.44a,b) and this implicitly defines the positive definite matrix $\mathcal{G}$ by

$$\hat{\underline{v}} \equiv \mathcal{G}\underline{v}. \tag{5.1.45}$$

It follows that $\mathcal{G}$ is the algebraic representation of $\widehat{\mathcal{G}}^h_N$. Noting (5.1.44a) and (5.1.45), we have

$$\mathcal{M}^{-1}\mathcal{K}\mathcal{G}\underline{v} = \underline{v}.$$

Substituting (5.1.43a,b) into (5.1.25a,b), noting (5.1), and then multiplying by $\mathcal{M}^{-1}\mathcal{K}\mathcal{M}^{-1}$. Also setting $\mathcal{R} \equiv \mathcal{M}^{-1}\mathcal{K}$, (5.1.25a,b) may be restated as find $\left\{\mathcal{U}_1^{k,j+1}, \mathcal{U}_2^{k,j+1}\right\} \in \mathbb{R}^{N\times 2} \times \mathbb{R}^{N\times 2}$ such that

$$\mathcal{R}\mathcal{U}_1^{k,j+1} + \mu\left(\gamma\mathcal{R}^2\mathcal{U}_1^{k,j+1} + \frac{1}{\Delta t}(\mathcal{U}_1^{k,j+1} - \mathcal{U}_1^{k-1}) + \frac{1}{2}\mathcal{R}D\mathcal{U}_2^{k,j+1}\right) = \mathcal{R}\mathcal{B}_1^{k,j+1},$$

and

$$\mathcal{R}\mathcal{U}_2^{k,j+1} + \mu\left(\gamma\mathcal{R}^2\mathcal{U}_2^{k,j+1} + \frac{1}{\Delta t}(\mathcal{U}_2^{k,j+1} - \mathcal{U}_2^{k-1}) + \frac{1}{2}\mathcal{R}D\mathcal{U}_1^{k,j+1}\right) = \mathcal{R}\mathcal{B}_2^{k,j+1}.$$

Systems of this type are solved using "the discrete cosine transform" when $\mathcal{T}^h$ is a uniform partitioning, see Barrett and Blowey [3].

Theorem 5.1.1 Let $\Delta t < \frac{4\gamma}{D^2}$, then for all $\mu \in \mathbb{R}^+$ and $\left\{U_1^{k,0}, U_2^{k,0}, W_1^{k,0}, W_2^{k,0}\right\} \in K^h_{m_1} \times K^h_{m_2} \times S^h \times S^h$ the sequence $\left\{U_1^{k,j}, U_2^{k,j}, W_1^{k,j}, W_2^{k,j}\right\}_{j\geq 0, i=1,2}$ generated by the algorithm (5.1.16a)-(5.1.19b) satisfies, as $j \to \infty$,

$$\|U_1^{k,j} - U_1^k\|_1^2 \longrightarrow 0 \text{ and } \int_\Omega |\nabla(W_1^{k,j+1} - W_1^k)|^2\,dx \longrightarrow 0, \tag{5.1.47a}$$

$$\|U_2^{k,j} - U_2^k\|_1^2 \longrightarrow 0 \text{ and } \int_\Omega |\nabla(W_2^{k,j+1} - W_2^k)|^2\,dx \longrightarrow 0. \tag{5.1.47b}$$

Proof: It follows from (5.1.14a,b), (5.1.15a,b), (5.1.16a,b), (5.1.18b), (5.1.19b) and by the definition of $X_1^{k,j+1}$ and $X_2^{k,j+1}$ that for $j \geq 0$

$$U_1^k = \frac{1}{2}(X_1^k + R_1^k), \qquad U_1^{k,j} = \frac{1}{2}(X_1^{k,j} + R_1^{k,j}), \qquad U_1^{k,j+\frac{1}{2}} = \frac{1}{2}(X_1^{k,j+1} + R_1^{k,j}) \tag{5.1.48a}$$

and

$$U_2^k = \frac{1}{2}(X_2^k + R_2^k), \qquad U_2^{k,j} = \frac{1}{2}(X_2^{k,j} + R_2^{k,j}), \qquad U_2^{k,j+\frac{1}{2}} = \frac{1}{2}(X_2^{k,j+1} + R_2^{k,j}). \tag{5.1.48b}$$

Noting (5.1.18b), (5.1.15a) with $\chi = U_1^{k,j+1} - U_1^k$ and (5.1.48a), we have

$$\begin{aligned}
&\gamma|U_1^{k,j+1} - U_1^k|_1^2 + \left(\frac{1}{2}D(U_2^{k,j+1} - U_2^k) - \ (W_1^{k,j+1} - W_1^k), U_1^{k,j+1} - U_1^k\right)^h \\
&= \ \frac{1}{\mu}\left[(X_1^{k,j+1} - X_1^k, U_1^{k,j+1} - U_1^k)^h - (U_1^{k,j+1} - U_1^k, U_1^{k,j+1} - U_1^k)^h\right] \\
&= \ \frac{1}{4\mu}(X_1^{k,j+1} - X_1^k - R_1^{k,j+1} + R_1^k, X_1^{k,j+1} - X_1^k + R_1^{k,j+1} - R_1^k)^h \\
&= \ \frac{1}{4\mu}\left(|X_1^{k,j+1} - X_1^k|_h^2 - |R_1^{k,j+1} - R_1^k|_h^2\right). \qquad (5.1.49a)
\end{aligned}$$

In the same way, noting (5.1.19b),(5.1.15b) and (5.1.48a) gives

$$\begin{aligned}
&\gamma|U_2^{k,j+1} - U_2^k|_1^2 + \left(\frac{1}{2}D(U_1^{k,j+1} - U_1^k) - \ (W_2^{k,p+1} - W_2^k), U_2^{k,j+1} - U_2^k\right)^h \\
&= \ \frac{1}{4\mu}\left(|X_2^{k,j+1} - X_2^k|_h^2 - |R_2^{k,j+1} - R_2^k|_h^2\right). \qquad (5.1.49b)
\end{aligned}$$

Choosing $\chi \equiv U_1^{k,j+\frac{1}{2}}$ in (5.1.12b) to obtain

$$(U_1^k, U_1^{k,j+\frac{1}{2}} - U_1^k)^h \ \geq \ (R_1^k, U_1^{k,j+\frac{1}{2}} - U_1^k)^h. \qquad (5.1.50)$$

For $m = 1 \to N$ choosing $r \equiv U_1^k(x_m)$ in (5.1.17a) multiplying by M_{jj} on recalling (3.1.2) and summing over j, we have that

$$(U_1^{k,j+\frac{1}{2}}, U_1^k - U_1^{k,j+\frac{1}{2}})^h \ \geq \ (R_1^{k,j}, U_1^k - U_1^{k,j+\frac{1}{2}})^h. \qquad (5.1.51)$$

Adding (5.1.50) and (5.1.51) gives

$$|U_1^{k,j+\frac{1}{2}} - U_1^k|_h^2 \ \leq \ (R_1^{k,j} - R_1^k, U_1^{k,j+\frac{1}{2}} - U_1^k)^h, \qquad (5.1.52)$$

then noting (5.1.48a) yields

$$|X_1^{k,j+1} - X_1^k|_h^2 \ \leq \ |R_1^{k,j} - R_1^k|_h^2.$$

As $U_1^{k,j+1}$, $U_1^k \in S_{m_1}^h$, choosing $\chi = \widehat{\mathcal{G}}_N^h\left(\frac{U_1^k - U_1^{k-1}}{\Delta t}\right) - \widehat{\mathcal{G}}_N^h\left(\frac{U_1^{k,j+1} - U_1^{k-1}}{\Delta t}\right)$ in (5.1.12a) and using (5.1.24a), we obtain that

$$\left(\frac{U_1^k - U_1^{k-1}}{\Delta t}, W_1^k - W_1^{k,j+1}\right)^h + (\nabla W_1^k, \nabla(W_1^k - W_1^{k,j+1})) \ = \ 0, \qquad (5.1.53)$$

and choosing $\chi = W_1^{k,j+1} - W_1^k$ in (5.1.18a) gives

$$\left(\frac{U_1^{k,j+1} - U_1^{k-1}}{\Delta t}, W_1^{k,j+1} - W_1^k\right)^h + (\nabla W_1^{k,j+1}, \nabla(W_1^{k,j+1} - W_1^k)) \ = \ 0, \quad (5.1.54)$$

then adding (5.1.53) and (5.1.54) it follows from (5.1.24a) that

$$-(W_1^{k,j+1}-W_1^k, U_1^{k,j+1}-U_1^k)^h = \Delta t\,|\nabla(W_1^{k,j+1}-W_1^k)|_0^2 = \frac{1}{\Delta t}\|U_1^{k,j+1}-U_1^k\|_{-h}^2. \tag{5.1.55}$$

Combining (5.1.49a), (5.1.52), (5.1.55) and rearranging yields that

$$\begin{aligned}
&\gamma|U_1^{k,j+1}-U_1^k|_1^2 \\
&+\frac{D}{2}(U_2^{k,j+1}-U_2^k, U_1^{k,j+1}-U_1^k)^h + \frac{1}{\Delta t}\|U_1^{k,j+1}-U_1^k\|_{-h} + \frac{1}{4\mu}|R_1^{k,j+1}-R_1^k|_h^2 \\
&\le \frac{1}{4\mu}|R_1^{k,j}-R_1^k|_h^2.
\end{aligned} \tag{5.1.56}$$

In the same way, choosing $\chi = U_2^{k,j+\frac{1}{2}}$ in (5.1.13b) and $r \equiv U_2^k(x_m)$ in (5.1.17b), we then have

$$|X_2^{k,j+1}-X_2^k|_h^2 \le |R_2^{k,j}-R_2^k|_h^2. \tag{5.1.57}$$

Choosing $\chi = W_2^k - W_2^{k,j+1}$ in (5.1.13a) and $\chi = W_2^{k,j+1} - W_2^k$ in (5.1.19a), then adding together to obtain

$$-(W_2^{k,j+1}-W_2^k, U_2^{k,j+1}-U_2^k)^h = \Delta t\,|\nabla(W_2^{k,j+1}-W_2^k)|_0^2 = \frac{1}{\Delta t}\|U_2^{k,j+1}-U_2^k\|_{-h}^2. \tag{5.1.58}$$

Combining (5.1.49b), (5.1.57) and (5.1.58), it follows that

$$\begin{aligned}
&\gamma|U_2^{k,j+1}-U_2^k|_1^2 \\
&+\frac{D}{2}(U_1^{k,j+1}-U_1^k, U_2^{k,j+1}-U_2^k)^h + \frac{1}{\Delta t}\|U_2^{k,j+1}-U_2^k\|_{-h} + \frac{1}{4\mu}|R_2^{k,j+1}-R_2^k|_h^2 \\
&\le \frac{1}{4\mu}|R_2^{k,j}-R_2^k|_h^2.
\end{aligned} \tag{5.1.59}$$

Noting that

$$\begin{aligned}
D|(U_1^{k,j+1}-U_1^k, U_2^{k,j+1}-U_2^k)^h| \le{}& \frac{1}{\Delta t}\Big(\|U_1^{k,j+1}-U_1^k\|_{-h}^2 + \|U_2^{k,j+1}-U_2^k\|_{-h}^2\Big) \\
&+\frac{\Delta t D^2}{4}\Big(|U_1^{k,j+1}-U_1^k|_1^2 + |U_2^{k,j+1}-U_2^k|_1^2\Big)
\end{aligned}$$

on combining this with (5.1.56) and (5.1.59) we have

$$\begin{aligned}
\left(\gamma - \frac{\Delta t D^2}{4}\right)\Big(|U_1^{k,j+1}-U_1^k|_1^2+|U_2^{k,j+1}-U_2^k|_1^2\Big)+\frac{1}{4\mu}\Big(|R_1^{k,j+1}-R_1^k|_h^2+|R_2^{k,j+1}-R_2^k|_h^2\Big) \\
\le \frac{1}{4\mu}\Big(|R_1^{k,j+1}-R_1^k|_h^2 + |R_2^{k,j+1}-R_2^k|_h^2\Big).
\end{aligned}$$

We have that $\left\{\frac{1}{4\mu}\left(|R_1^{k,j}-R_1^k|_h^2+|R_2^{k,j}-R_2^k|_h^2\right)\right\}_{j\geq 0}$ is a decreasing sequence which is bounded below and so has a limit.

Therefore, letting $j\to\infty$ we find that

$$\lim_{j\to\infty}\left(\gamma-\frac{\Delta t D^2}{4}\right)\left(|U_1^{k,j+1}-U_1^k|_1^2+|U_2^{k,j+1}-U_2^k|_1^2\right)\ \leq\ 0,$$

so that as $U_i^{k,j+1}-U_i^k\in S_0^h$ and $\Delta t<\frac{4\gamma}{D^2}$, (5.1.47a,b) follows on noting (5.1.55) and (5.1.58). □

5.2 Linear Stability Analysis: One Dimensional Case

We consider a coupled pair of Cahn-Hilliard equations in one dimension where we assume that $|u_i(x,t)|<1$, for $i=1,2$,

Find $\{u_1(x,t),u_2(x,t)\}\in\mathbb{R}\times\mathbb{R}$ such that, for $i=1,2$,

$$\frac{\partial u_i}{\partial t} = (w_i)_{xx} \qquad \Omega=(0,1),\ t>0, \tag{5.2.1a}$$

$$w_i = -\gamma(u_i)_{xx}-u_i+D(u_j+1), \tag{5.2.1b}$$

where

$$\frac{\partial u_i}{\partial x} = \frac{\partial w_i}{\partial x} = 0 \qquad \text{on } \partial\Omega, \tag{5.2.1c}$$

$$u_i(x,0) = u_i^0(x) \qquad \text{on } \Omega. \tag{5.2.1d}$$

We assume the solution of the linearised problem is of the form

$$u_i(x,t) = \frac{m_i}{|\Omega|}+\sum_{n=1}^{\infty}\cos(n\pi x)F_{i,n}(t), \tag{5.2.2}$$

so that (5.2.1c) is automatically satisfied. Then we have

$$D(u_j+1) = D\left(\frac{m_j}{|\Omega|}+\sum_{n=1}^{\infty}\cos(n\pi x)F_{j,n}(t)+1\right). \tag{5.2.3}$$

Substituting (5.2.3) into (5.2.1b) and then into (5.2.1a), multiplying both sides by $\cos(m\pi x)$, $m\geq 1$, then integrating over the interval $(0,1)$ to give

$$\frac{dF_{i,m}(t)}{dt} = -\gamma n^4\pi^4F_{i,m}(t)+n^2\pi^2F_{i,m}(t)-Dn^2\pi^2F_{j,m}(t),\ i\neq j,$$

since

$$\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \frac{1}{2} & \text{for } m = n \leq 1, \end{cases}$$

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{m_i}{|\Omega|} + \sum_{n=1}^{\infty}\cos(n\pi x)F_{i,n}(t)\right) \\ &= \sum_{n=1}^{\infty}\cos(n\pi x)\frac{d}{dt}F_{i,n}(t), \end{aligned}$$

$$\begin{aligned} (-\gamma(u_i)_{xx})_{xx} &= \left(-\gamma\left(\frac{m_i}{|\Omega|} + \sum_{n=1}^{\infty}\cos(n\pi x)F_{i,n}(t)\right)_{xx}\right)_{xx} \\ &= \left(-\gamma\left(-n^2\pi^2\sum_{n=1}^{\infty}\cos(n\pi x)F_{i,n}(t)\right)\right)_{xx} \\ &= -\gamma n^4\pi^4\sum_{n=1}^{\infty}\cos(n\pi x)F_{i,n}(t), \end{aligned}$$

$$\begin{aligned} -(u_i(x,t))_{xx} &= \left(-\sum_{n=1}^{\infty}\cos(n\pi x)F_{i,n}(t)\right)_{xx} \\ &= n^2\pi^2\sum_{n=1}^{\infty}\cos(n\pi x)F_{i,n}(t), \end{aligned}$$

and

$$\begin{aligned} D(u_j+1)_{xx} &= D\left(\frac{m_j}{|\Omega|} + \sum_{n=1}^{\infty}\cos(n\pi x)F_{j,n}(t) + 1\right)_{xx} \\ &= -Dn^2\pi^2\sum_{n=1}^{\infty}\cos(n\pi x)F_{j,n}(t). \end{aligned}$$

In term of vectors we express as

$$\frac{d}{dt}\begin{pmatrix} F_{1,m}(t) \\ F_{2,m}(t) \end{pmatrix} = \begin{pmatrix} -\gamma m^4\pi^4 + m^2\pi^2 & -Dm^2\pi^2 \\ -Dm^2\pi^2 & -\gamma m^4\pi^4 + m^2\pi^2 \end{pmatrix}\begin{pmatrix} F_{1,m}(t) \\ F_{2,m}(t) \end{pmatrix}.$$

Thus the solution is given by

$$\begin{pmatrix} F_{1,m}(t) \\ F_{2,m}(t) \end{pmatrix} = \exp(tA)\begin{pmatrix} F_{1,m}(0) \\ F_{2,m}(0) \end{pmatrix},$$

where

$$A = \begin{pmatrix} -\gamma m^4\pi^4 + m^2\pi^2 & -Dm^2\pi^2 \\ -Dm^2\pi^2 & -\gamma m^4\pi^4 + m^2\pi^2 \end{pmatrix}.$$

A simple calculation gives that the eigenvalues of the matrix A are

$$\lambda_1 = -\gamma m^4\pi^4 + m^2\pi^2 + Dm^2\pi^2 \quad \text{and} \quad \lambda_2 = -\gamma m^4\pi^4 + m^2\pi^2 - Dm^2\pi^2.$$

The corresponding eigenvectors of λ_1 and λ_2 are

$$e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{respectively.}$$

Thus we obtain that

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

where $a = -\gamma n^4\pi^4 + n^2\pi^2$, and $b = -Dn^2\pi^2$.

Noting $\exp(At) = \sum_{j=0}^{\infty} \frac{t^j}{j!}A^j$ yield

$$\begin{aligned} \exp(At) &= \sum_{j=0}^{\infty} \frac{t^j}{j!}\left[\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}^j\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \exp((a-b)t) & 0 \\ 0 & \exp((a+b)t) \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

Hence we have that

$$\begin{aligned} &\begin{pmatrix} F_{1,m}(t) \\ F_{2,m}(t) \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \exp((a-b)t) & 0 \\ 0 & \exp((a+b)t) \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} F_{1,m}(0) \\ F_{2,m}(0) \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \exp((a-b)t) & 0 \\ 0 & \exp((a+b)t) \end{pmatrix}\begin{pmatrix} F_{1,m}(0) - F_{2,m}(t) \\ F_{2,m}(0) + F_{2,m}(0) \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \exp((a-b)t)\Big(F_{1,m}(0) - F_{2,m}(0)\Big) \\ \exp((a+b)t)\Big(F_{1,m}(0) + F_{2,m}(0)\Big) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\Big[\exp((a-b)t)\Big(F_{1,m}(0) - F_{2,m}(0)\Big) + \exp((a+b)t)\Big(F_{1,m}(0) + F_{2,m}(0)\Big)\Big] \\ \frac{1}{2}\Big[-\exp((a-b)t)\Big(F_{1,m}(0) - F_{2,m}(0)\Big) + \exp((a+b)t)\Big(F_{1,m}(0) + F_{2,m}(0)\Big)\Big] \end{pmatrix}. \end{aligned}$$

Note that there is no m_1 and m_2 dependence on whether or not there is growth, cf. Imran [33]. Therefore we obtain that

$$\begin{aligned} u_1(x,t) \; = \; \frac{m_1}{|\Omega|}+\frac{1}{2}\sum_{n=1}^{\infty}\cos(n\pi x)\Big[\exp((a-b)t)\Big(F_{1,n}(0)-F_{2,n}(0)\Big) \\ + \exp((a+b)t)\Big(F_{1,n}(0)+F_{2,n}(0)\Big)\Big], \end{aligned} \tag{5.2.4a}$$

and

$$\begin{aligned} u_2(x,t) \; = \; \frac{m_2}{|\Omega|}+\frac{1}{2}\sum_{n=1}^{\infty}\cos(n\pi x)\Big[-\exp((a-b)t)\Big(F_{1,n}(0)-F_{2,n}(0)\Big) \\ + \exp((a+b)t)\Big(F_{1,n}(0)+F_{2,m}(0)\Big)\Big]. \end{aligned} \tag{5.2.4b}$$

5.3 Numerical Simulations

5.3.1 One Dimensional Case

Numerical simulations in one dimension were performed using the decoupled ($\beta = 0$) and coupled ($\beta = \frac{1}{2}$) schemes with $\Omega = (0,1)$. In all simulations we take $\gamma = 0.005$, $D = 0.5$ and $TOL = 1 \times 10^{-10}$.

A comparison between the solution of linearised problem and the numerical approximation

We consider the problem (5.2.1a-d) with the following initial conditions

$$u_i(x,0) \; = \; u_i^0(x) \; = \; \zeta_i \cos(\pi x), \quad \text{for} \;\; i = 1,2,$$

where the ζ_i are small.

Setting $\zeta_1 = \zeta_2$ then comparing to (5.2.2), we have that

$$n \; = \; 1, \quad m_i \; = \; 0, \quad F_{i,n}(0) \; = \; \zeta_i \quad \text{and} \quad F_{i,k} = 0 \quad \text{for} \;\; k \neq 1, i = 1,2. \tag{5.3.1}$$

Hence on noting (5.2.4a,b) the solution of the linearised problem is

$$u_i(x,t) = \zeta_1 \cos(\pi x)\exp((a+b)t),$$

where $a = -\gamma\pi^4 + \pi^2$ and $b = -D\pi^2$.

If we let $\zeta_1 = \frac{\zeta_2}{2}$ while keeping the other parameters as in (5.3.1), then

$$F_{1,n}(0) = \zeta_1 \quad \text{and} \quad F_{2,n}(0) = 2\zeta_1$$

and the solution (5.2.4a,b) becomes

$$\begin{aligned} u_1(x,t) &= \frac{\zeta_1}{2}\cos(\pi x)\Big[-\exp((a-b)t) + 3\exp((a+b)t)\Big], \\ u_2(x,t) &= \frac{\zeta_1}{2}\cos(\pi x)\Big[\exp((a-b)t) + 3\exp((a+b)t)\Big]. \end{aligned}$$

In this section we present five experiments in one dimension where the decoupled ($\beta = 0$) and coupled ($\beta = \frac{1}{2}$) schemes have been solved separately.

In the first experiment we set $F_{1,1} = F_{2,1} = 0.001$. The maximum errors at $t = 0.25$ with various h and Δt have been shown in Table 5.1 and Table 5.2 for the decoupled scheme, and Table 5.3 and Table 5.4 for the coupled scheme. We see that the maximum errors decrease approximately linearly with Δt when Δt is proportional to h and also when Δt is proportional to h^2. The comparison between numerical solutions and linear stability analysis solutions for the decoupled scheme have been depicted in Figure 5.1. We can see that in the numerical solution rounding errors have led to the introduction of a $\cos(3\pi x)$ component which corresponds to a faster growth rate than that of the $\cos(\pi x)$ component.

In the second and third experiments the initial condition are given by taking $F_{1,1} = \frac{F_{2,1}}{2}$, $F_{2,1} = 0.002$ and $F_{1,1} = 2F_{2,1}$, $F_{2,1} = 0.001$. The maximum errors for both experiments have been respectively shown in Table 5.5 - Table 5.8 and Table 5.9 - Table 5.12. We found that the numerical solutions for both experiments were symmetric.

h^{-1}	$(\Delta t)^{-1}$	$\lVert u_1 - U_1 \rVert_\infty$	$\lVert u_2 - U_2 \rVert_\infty$
32	256	$5.87798382 \times 10^{-5}$	$1.99423955 \times 10^{-4}$
64	512	$3.29621298 \times 10^{-5}$	$1.02682757 \times 10^{-4}$
128	1024	$1.73478005 \times 10^{-5}$	$5.21448920 \times 10^{-5}$
256	2048	$8.85900830 \times 10^{-6}$	$2.63089470 \times 10^{-5}$
512	4096	$4.47220320 \times 10^{-6}$	$1.32180250 \times 10^{-5}$

Table 5.1: Maximum error at $t = 0.25$, where $\beta = 0$, $F_{1,n} = F_{2,n} = 0.001$, $\Delta t = \frac{1}{8}h$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

h^{-1}	$(\Delta t)^{-1}$	$\lVert u_1 - U_1 \rVert_\infty$	$\lVert u_2 - U_2 \rVert_\infty$
32	256	$5.87798382 \times 10^{-5}$	$1.99423955 \times 10^{-4}$
64	1024	$1.68791947 \times 10^{-5}$	5.2568148×10^{-5}
128	4096	$4.32916580 \times 10^{-6}$	1.3357218×10^{-5}

Table 5.2: Maximum error at $t = 0.25$, where $\beta = 0$, $F_{1,n} = F_{2,n} = 0.001$, $\Delta t = 4h^2$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

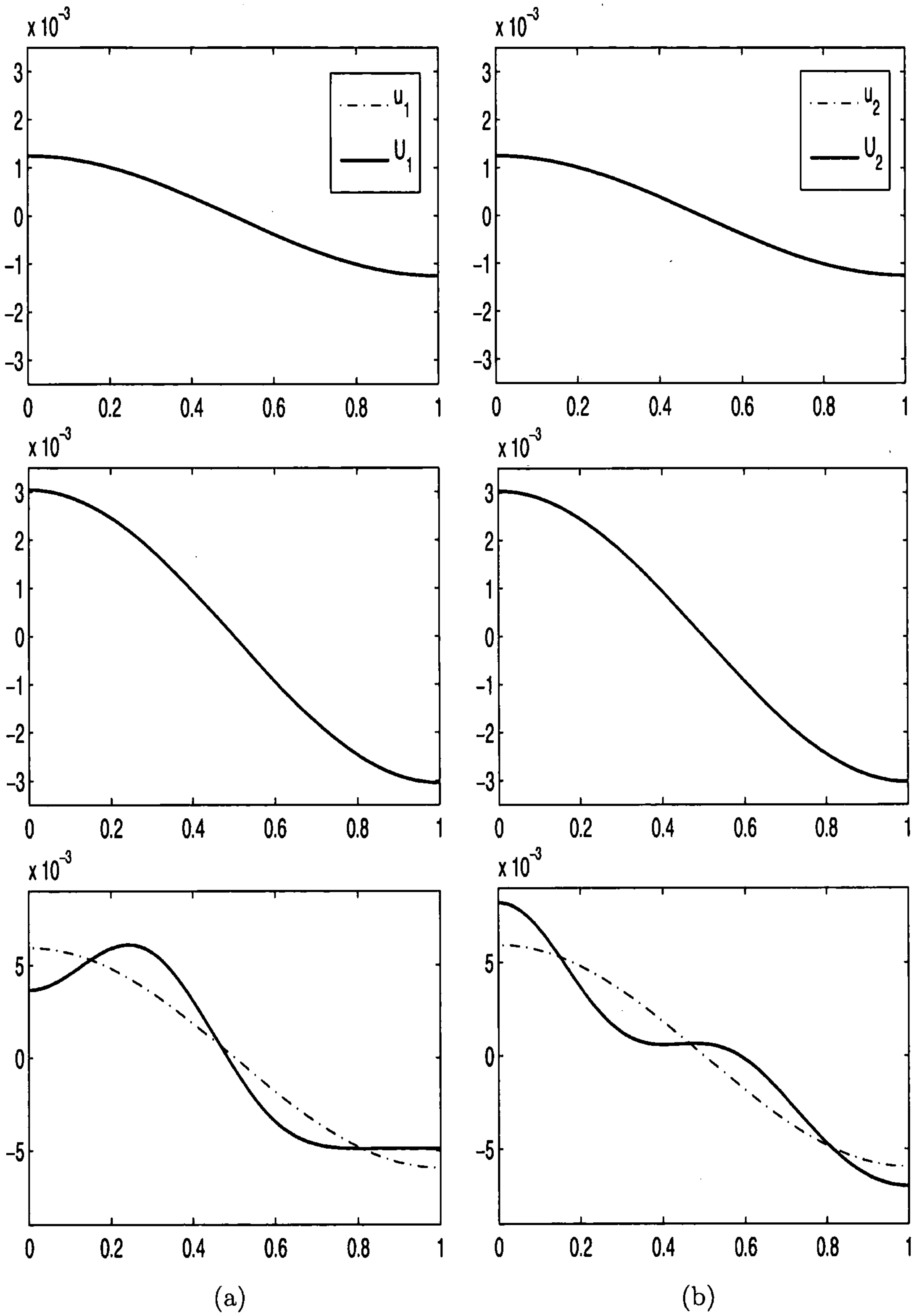


Figure 5.1: Numerical solution U_i and linear stability analysis solution u_i for $(a)\, i = 1$, $(b)\, i = 2$ where $\beta = 0$, $F_{1,n} = F_{2,n} = 0.001$, $n = 1$, and $t = 0.05, 0.25, 0.40$.

h^{-1}	$(\Delta t)^{-1}$	$\|u_1 - U_1\|_\infty$	$\|u_2 - U_2\|_\infty$
32	256	$6.13660450 \times 10^{-5}$	$6.13660450 \times 10^{-5}$
64	512	$3.06747934 \times 10^{-5}$	$3.06747934 \times 10^{-5}$
128	1024	$1.54797199 \times 10^{-5}$	$1.54797199 \times 10^{-5}$
256	2048	$7.36275210 \times 10^{-6}$	$7.36275210 \times 10^{-6}$
512	4096	$4.06392750 \times 10^{-6}$	$4.06392750 \times 10^{-6}$

Table 5.3: Maximum error at $t = 0.25$, where $\beta = \frac{1}{2}$, $F_{1,n} = F_{2,n} = 0.001$, $\Delta t = \frac{1}{8}h$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

h^{-1}	$(\Delta t)^{-1}$	$\|u_1 - U_1\|_\infty$	$\|u_2 - U_2\|_\infty$
32	256	$6.13660450 \times 10^{-5}$	$6.13660450 \times 10^{-5}$
64	1024	$1.49337038 \times 10^{-5}$	$1.49337038 \times 10^{-5}$
128	4096	$4.03477361 \times 10^{-6}$	$4.03477361 \times 10^{-6}$

Table 5.4: Maximum error at $t = 0.25$, where $\beta = \frac{1}{2}$, $F_{1,n} = F_{2,n} = 0.001$, $\Delta t = 4h^2$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

h^{-1}	$(\Delta t)^{-1}$	$\|u_1 - U_1\|_\infty$	$\|u_2 - U_2\|_\infty$
32	256	$1.57448860 \times 10^{-3}$	$1.37118948 \times 10^{-3}$
64	512	$8.57714221 \times 10^{-4}$	$7.48126182 \times 10^{-4}$
128	1024	$5.54432867 \times 10^{-4}$	$5.00774901 \times 10^{-4}$
256	2048	$4.70444270 \times 10^{-4}$	$4.45612504 \times 10^{-4}$
512	4096	$4.62104810 \times 10^{-4}$	$4.50626816 \times 10^{-4}$

Table 5.5: Maximum error at $t = 0.25$, where $\beta = 0$, $F_{1,n} = 0.001$, $F_{2,n} = 0.002$, $\Delta t = \frac{1}{8}h$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

h^{-1}	$(\Delta t)^{-1}$	$\|u_1 - U_1\|_\infty$	$\|u_2 - U_2\|_\infty$
32	256	$1.57448860 \times 10^{-3}$	$1.37118948 \times 10^{-3}$
64	1024	$5.62485313 \times 10^{-4}$	$5.10229662 \times 10^{-4}$
128	4096	$4.64564336 \times 10^{-4}$	$4.53513768 \times 10^{-4}$

Table 5.6: Maximum error at $t = 0.25$, where $\beta = 0$, $F_{1,n} = 0.001$, $F_{2,n} = 0.002$, $\Delta t = 4h^2$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

h^{-1}	$(\Delta t)^{-1}$	$\|u_1 - U_1\|_\infty$	$\|u_2 - U_2\|_\infty$
32	256	$3.16815893 \times 10^{-3}$	$3.27650411 \times 10^{-3}$
64	512	$1.47401926 \times 10^{-3}$	$1.52915157 \times 10^{-3}$
128	1024	$1.02015096 \times 10^{-3}$	$1.04773149 \times 10^{-3}$
256	2048	$8.61216104 \times 10^{-4}$	$8.74146353 \times 10^{-4}$
512	4096	$7.82690186 \times 10^{-4}$	$7.89781450 \times 10^{-4}$

Table 5.7: Maximum error at $t = 0.25$, where $\beta = \frac{1}{2}$, $F_{1,n} = 0.001$, $F_{2,n} = 0.002$, $\Delta t = \frac{1}{8}h$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

h^{-1}	$(\Delta t)^{-1}$	$\|u_1 - U_1\|_\infty$	$\|u_2 - U_2\|_\infty$
32	256	$3.16815893 \times 10^{-3}$	$3.27650411 \times 10^{-3}$
64	1024	$1.01388805 \times 10^{-3}$	$1.04085228 \times 10^{-3}$
128	4096	$7.80084636 \times 10^{-4}$	$7.87188765 \times 10^{-4}$

Table 5.8: Maximum error at $t = 0.25$, where $\beta = \frac{1}{2}$, $F_{1,n} = 0.001$, $F_{2,n} = 0.002$, $\Delta t = 4h^2$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

h^{-1}	$(\Delta t)^{-1}$	$\|u_1 - U_1\|_\infty$	$\|u_2 - U_2\|_\infty$
32	256	$1.39806788 \times 10^{-3}$	$7.74157991 \times 10^{-4}$
64	512	$7.59569017 \times 10^{-4}$	$4.42124777 \times 10^{-4}$
128	1024	$5.03354958 \times 10^{-4}$	$3.46448602 \times 10^{-4}$
256	2048	$4.45266735 \times 10^{-4}$	$3.68990717 \times 10^{-4}$
512	4096	$4.49211703 \times 10^{-4}$	$4.12042962 \times 10^{-4}$

Table 5.9: Maximum error at $t = 0.25$, where $\beta = 0$, $F_{1,n} = 0.002$, $F_{2,n} = 0.001$, $\Delta t = \frac{1}{8}h$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

h^{-1}	$(\Delta t)^{-1}$	$\|u_1 - U_1\|_\infty$	$\|u_2 - U_2\|_\infty$
32	256	$1.39806788 \times 10^{-3}$	$7.74157991 \times 10^{-4}$
64	1024	$5.12957947 \times 10^{-4}$	$3.54784476 \times 10^{-4}$
128	4096	$4.52051128 \times 10^{-4}$	$4.14462769 \times 10^{-4}$

Table 5.10: Maximum error at $t = 0.25$, where $\beta = 0$, $F_{1,n} = 0.002$, $F_{2,n} = 0.001$, $\Delta t = 4h^2$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

h^{-1}	$(\Delta t)^{-1}$	$\|u_1 - U_1\|_\infty$	$\|u_2 - U_2\|_\infty$
32	256	$3.27650411 \times 10^{-3}$	$3.16815893 \times 10^{-3}$
64	512	$1.52915157 \times 10^{-3}$	$1.47401926 \times 10^{-3}$
128	1024	$1.04773149 \times 10^{-3}$	$1.02015096 \times 10^{-3}$
256	2048	$8.74146353 \times 10^{-4}$	$8.61216104 \times 10^{-4}$
512	4096	$7.89781450 \times 10^{-4}$	$7.82690186 \times 10^{-4}$

Table 5.11: Maximum error at $t = 0.25$, where $\beta = \frac{1}{2}$, $F_{1,n} = 0.002$, $F_{2,n} = 0.001$, $\Delta t = \frac{1}{8}h$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

h^{-1}	$(\Delta t)^{-1}$	$\|u_1 - U_1\|_\infty$	$\|u_2 - U_2\|_\infty$
32	256	$3.27650411 \times 10^{-3}$	$3.16815893 \times 10^{-3}$
64	1024	$1.04085228 \times 10^{-3}$	$1.01388805 \times 10^{-3}$
128	4096	$7.87188765 \times 10^{-4}$	$7.80084636 \times 10^{-4}$

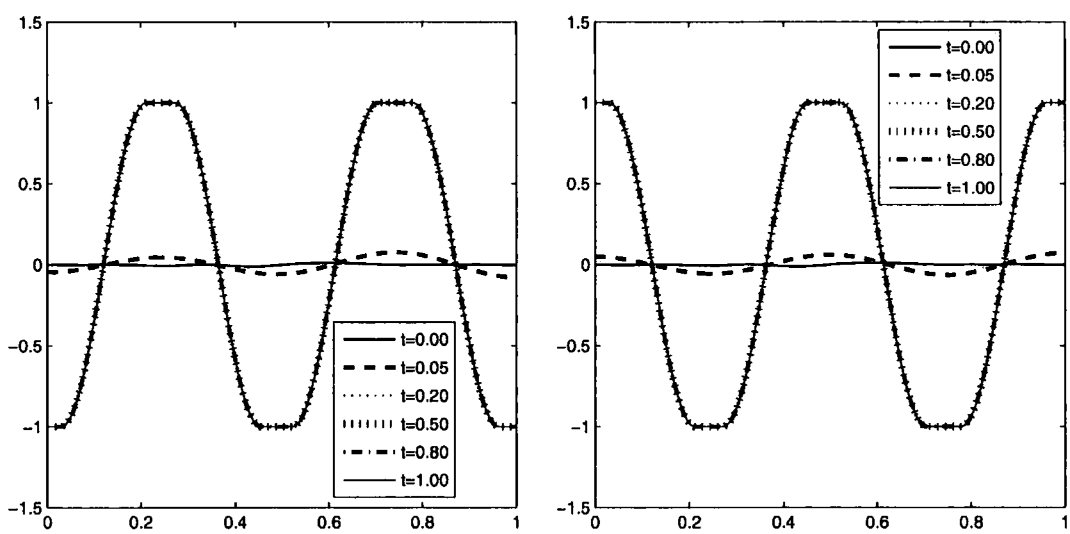
Table 5.12: Maximum error at $t = 0.25$, where $\beta = \frac{1}{2}$, $F_{1,n} = 0.002$, $F_{2,n} = 0.001$, $\Delta t = 4h^2$, $\gamma = 0.005$, $n = 1$, $D = 0.5$ with $TOL = 1 \times 10^{-10}$.

Simulations with no exact solutions

In this section we consider the initial condition

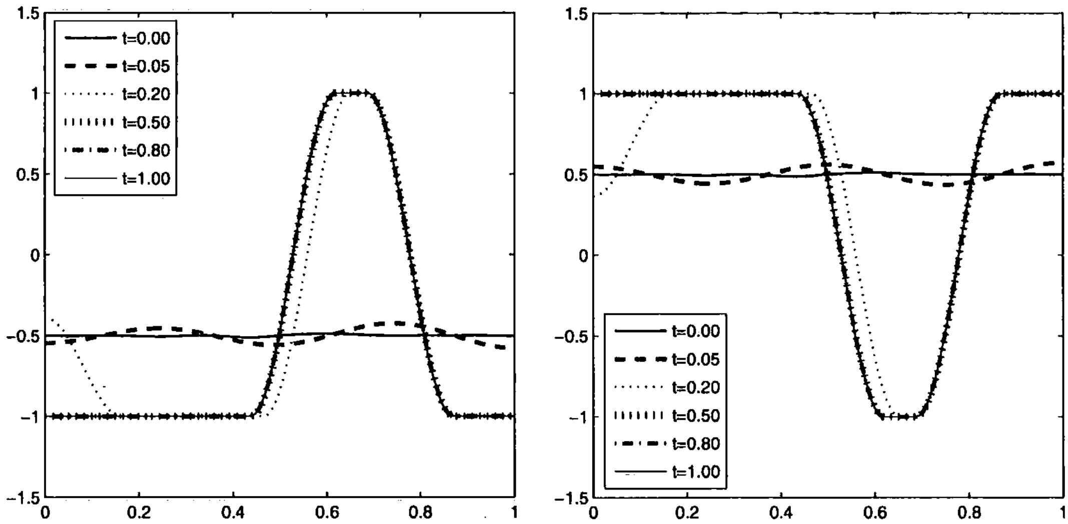
$$U_i^0 = U_i^m + \zeta(x), \tag{5.3.3}$$

where $\zeta(x)$ is a random perturbation of the state $U = 0$ with values distributed uniformly between -0.05 and 0.05. We performed seven experiments for both decoupled and coupled schemes by setting $h = \frac{1}{128}$, $\Delta t = \frac{h}{32}$ and taking different (m_1, m_2) and random initial values. We have noticed that for each choice of (m_1, m_2), Figure 5.2 - Figure 5.8, the stationary solutions of the decoupled and coupled schemes are the same so we only present the $\beta = \frac{1}{2}$ case. Moreover, the approximate solutions are also in between -1 and 1. Where possible all numerical stationary solution consists of many spatial intervals where $(U_1, U_2) = (1, -1), (-1, 1)$ or $(-1, -1)$. What appears to be the case is that $(1, 1)$ is avoided.



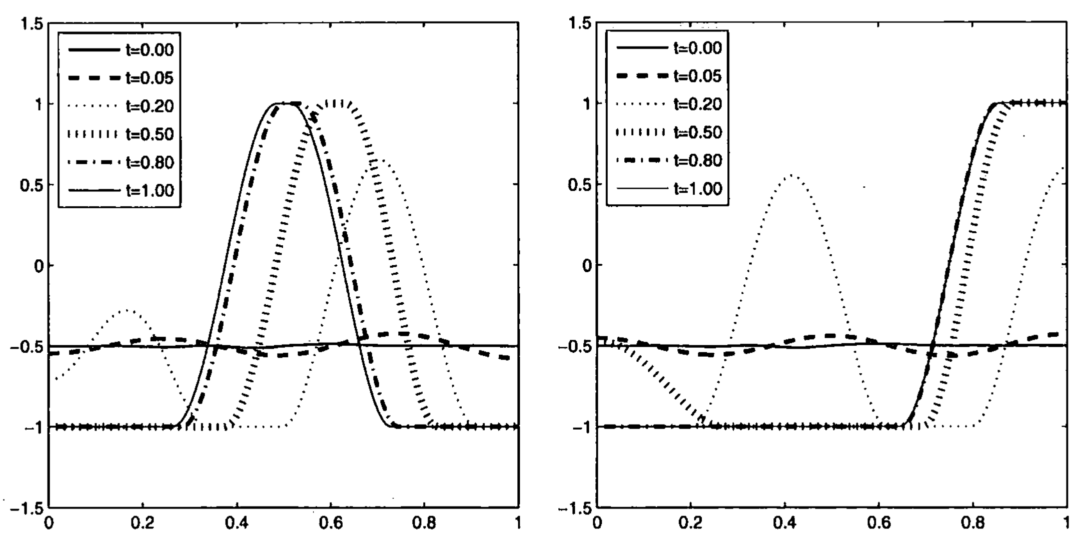
(a) U_1 (b) U_2

Figure 5.2: Numerical solution where $\beta = \frac{1}{2}$, $m_1 = m_2 = 0.00$, and $t = 0.00$, 0.05, 0.20, 0.50, 0.80, 1.00.



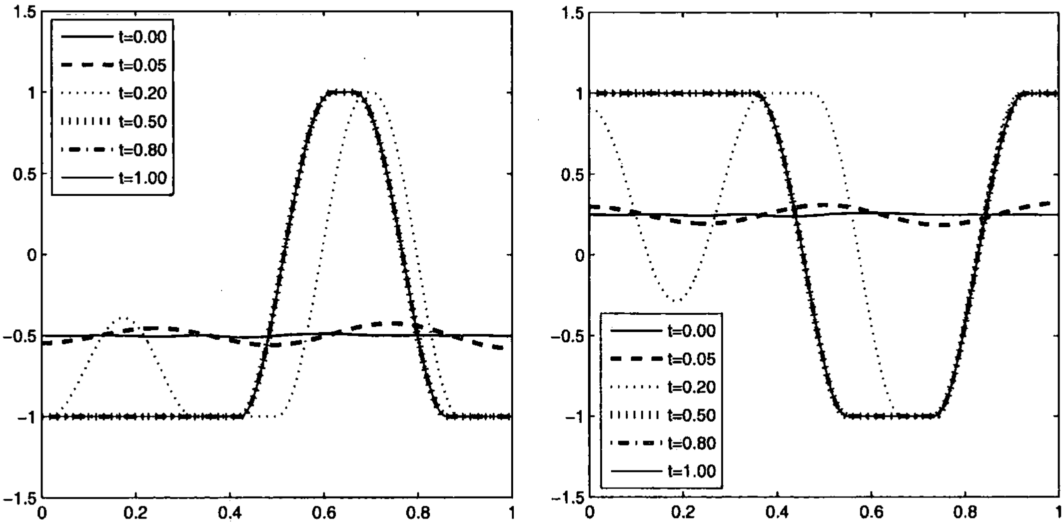
(a) U_1 (b) U_2

Figure 5.3: Numerical solution where $\beta = \frac{1}{2}$, $m_1 = -0.50$, $m_2 = 0.50$, and $t = 0.00, 0.05, 0.20, 0.50, 0.80, 1.00$.



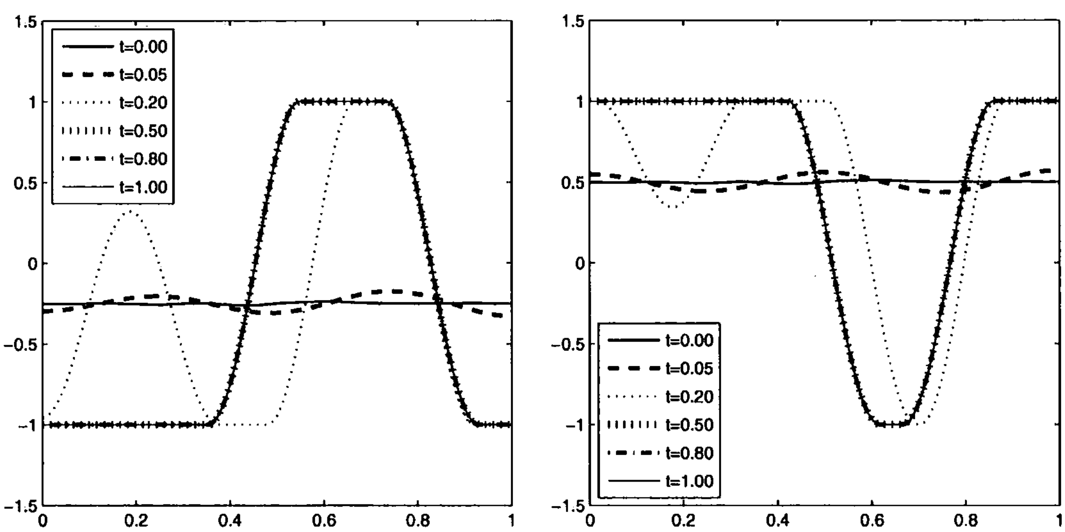
(a) U_1 (b) U_2

Figure 5.4: Numerical solution where $\beta = \frac{1}{2}$, $m_1 = -0.50$, $m_2 = -0.50$, and $t = 0.00, 0.05, 0.20, 0.50, 0.80, 1.00$.



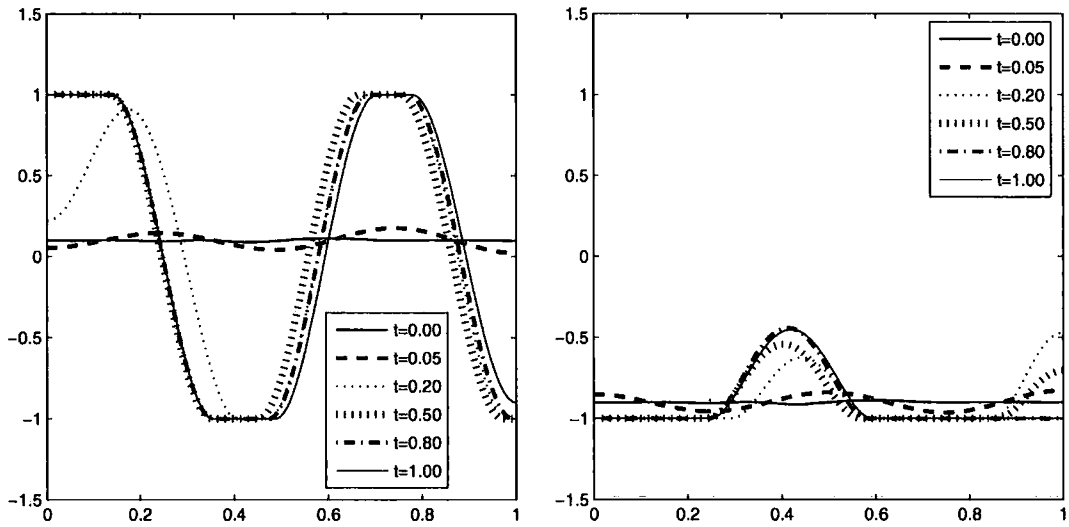
(a) U_1 (b) U_2

Figure 5.5: Numerical solution where $\beta = \frac{1}{2}$, $n = 1$, $m_1 = -0.50$, $m_2 = 0.25$, and $t = 0.00, 0.05, 0.20, 0.50, 0.80, 1.00$.



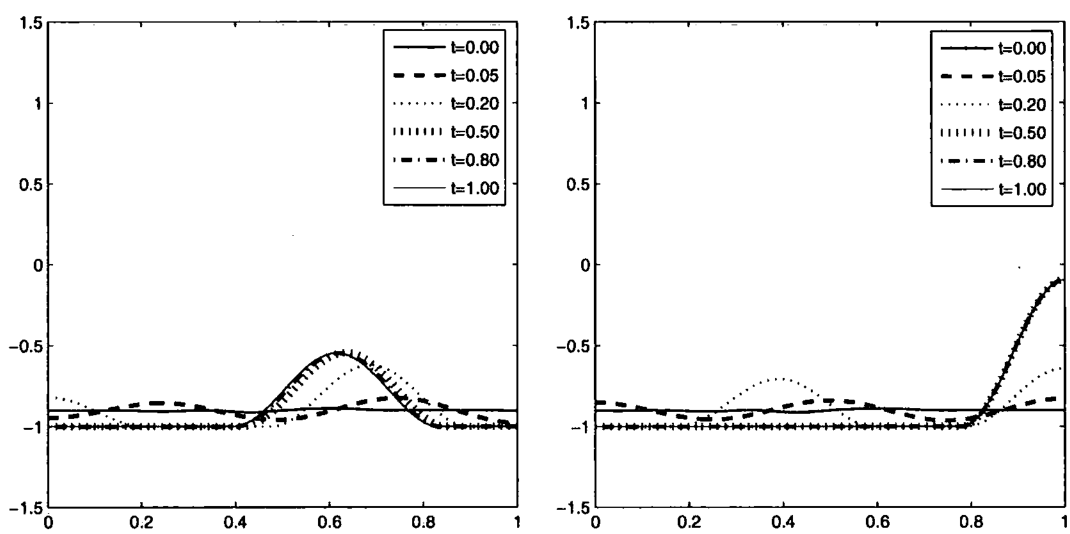
(a) U_1 (b) U_2

Figure 5.6: Numerical solution where $\beta = \frac{1}{2}$, $m_1 = -0.25$, $m_2 = 0.50$, and $t = 0.00, 0.05, 0.20, 0.50, 0.80, 1.00$.



(a) U_1 (b) U_2

Figure 5.7: Numerical solution where $\beta = \frac{1}{2}$, $m_1 = 0.10$, $m_2 = -0.90$, and $t = 0.00, 0.05, 0.20, 0.50, 0.80, 1.00$.



(a) U_1 (b) U_2

Figure 5.8: Numerical solution where $\beta = \frac{1}{2}$, $m_1 = -0.90$, $m_2 = -0.90$, and $t = 0.00, 0.05, 0.20, 0.50, 0.80, 1.00$.

We have shown that $\mathcal{E}_\gamma(u_1, u_2)$ is a Lyapunov functional and so it is natural to consider the minimization problem (M_γ):

$$\inf_{(u_1,u_2)\,\in\, K_{m_1}\times K_{m_2}} \mathcal{E}_\gamma(u_1, u_2)$$

which has stationary solutions given by:
Find $u_i \in K_{m_i}$ and $\lambda_i \in \mathbb{R}$ such that for all $\eta \in K$

$$\gamma(\nabla u_i, \nabla\eta - \nabla u_i) - (u_i, \eta - u_i) + D(u_j + 1, \eta - u_i) \;\geq\; \lambda_i(1, \eta - u_i), i = 1, 2.$$

We have exhibited several numerical approximations to these solutions in our numerical experiments which will satisfy the problem:

Find $U_i \in K^h_{m_i}$ and $\lambda^h_i \in \mathbb{R}$ such that for all $\eta^h \in K^h$

$$\gamma(\nabla U_i, \nabla\eta^h - \nabla U_i) - (U_i, \eta^h - U_i)^h + D(U_j + 1, \eta^h - U_i)^h \;\geq\; \lambda^h_i(1, \eta^h - U_i)^h, i = 1, 2.$$

Consideration of the asymptotic behaviour as $\gamma \to 0$ of (M_γ) as considered by Modica [40] and Luckhaus and Modica [39] is beyond the scope of this thesis. However, we formally consider the problem

$$\min \int_\Omega \psi(u_1, u_2)\, dx \;=\; \int_\Omega \frac{1 - u_1^2}{2} + \frac{1 - u_2^2}{2} + D(1 + u_1)(1 + u_2)\, dx,$$

where

$$|u_i| \;\leq\; 1 \quad \text{and} \quad \int_\Omega u_i \;=\; m_i.$$

It is clear that $\psi(u_1, u_2) \geq 0$ and as ψ has three minima at

$$(-1, -1), \quad (-1, 1), \quad (1, -1).$$

The solution of this problem is

$$u_1 \;=\; \begin{cases} 1 & \text{in } \Omega^1_+, \\ -1 & \text{in } \Omega^1_-, \end{cases} \qquad \text{and} \qquad u_2 \;=\; \begin{cases} 1 & \text{in } \Omega^2_+, \\ -1 & \text{in } \Omega^2_-, \end{cases}$$

where $\Omega^1_+ \cap \Omega^2_+ = \emptyset$, $\bar{\Omega} = \bar{\Omega}^1_+ \cup \bar{\Omega}^1_- = \bar{\Omega}^2_+ \cup \bar{\Omega}^2_-$, such that

$$|\Omega| m_1 \;=\; |\Omega^1_+| - |\Omega^1_-|, \qquad |\Omega| m_2 \;=\; |\Omega^2_+| - |\Omega^2_-|$$

if regions $\Omega^i_\pm$ exist.

If one were to attempt to generalize the results of Luckhuas and Modica [39] the geodesics connecting the minima are key. It is interesting to calculate

$$\int_\Gamma \psi(u_1, u_2)\, ds,$$

where

$$\Gamma_1 := (-1, 1-2t), \qquad \Gamma_2 := (1-2t, -1), \qquad \Gamma_3 := (-1+2t, 1-2t),$$

for $t \in [0,1]$ which corresponds to the straight line paths connecting the three minima.

We find that

$$\begin{aligned}\int_{\Gamma_1} \psi(u_1, u_2) ds &= \int_0^1 2\psi(-1, 1-2t)\, dt \\ &= \int_0^1 2 \times \frac{1-(1-2t)^2}{2}\, dt \\ &= \int_0^1 4(t-t^2)\, dt = 2 - \frac{4}{3} = \frac{2}{3}\end{aligned}$$

and similarly

$$\int_{\Gamma_2} \psi(u_1, u_2) ds = \frac{2}{3}$$

while

$$\int_{\Gamma_3} \psi(u_1, u_2) ds = \frac{4}{3}(1+D).$$

This lead us to the tentative conclusion, which is demonstrated in two dimensional simulations that where two close regions consist of $(-1,1)$ and $(1,-1)$ there will be a thin layer of $(-1,-1)$.

5.3.2 Two Dimensional Case

In the two dimension simulations we consider the $N \times N$ uniform square mesh on Ω. We then choose a particular right-angled triangulation of Ω in which each subsquare of length h is divided by the north east diagonal, see Figure 5.9.

Five separate simulations are performed in this section. We will always obtain phase separation due to $[-1,1]^2$ being the global spinodal region for γ sufficiently

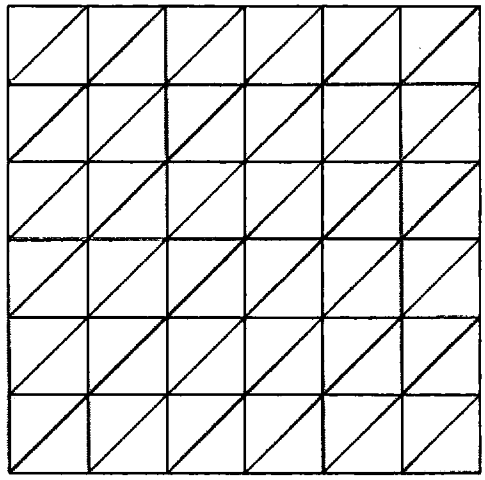


Figure 5.9: Uniform mesh with right-angled triangulation in each subsquare

small as observed by the linear stability analysis for one space dimension. In each simulation we set $\gamma = 0.005$, $\Omega = (0,1)^2$, $h = \frac{1}{64}$, $\Delta t = \frac{1}{512}$, $D = 0.5$, $\mu = 0.001$ and $TOL = 1 \times 10^{-8}$. The initial conditions were taken to be a random perturbation, with values distributed between -0.05 and 0.05, of a uniform state (m_1, m_2).

In order to make the graphical representation of the solution U_1 and U_2 on the mesh easier, we map the pointwise values of the two variables into an RGB colour. Fix i and j, then we take the values of U_1 and U_2 at the points (x_i, y_j), (x_i, y_{j+1}), (x_{i+1}, y_j),(x_{i+1}, y_{j+1}), and average, giving us two values s_1 and s_2. We then define the RGB colour to be

$$\left(\frac{1}{2}(1+s_1), \frac{1}{2}(1+s_2), 1 - \frac{1}{2}(s_1+1) - \frac{1}{2}(s_2+1)\right)$$

which has the property that when $(s_1, s_2) = (1,-1)$, $(1,-1)$ and $(-1,-1)$ we get pure red, green and blue. These values correspond to the minimum of

$$\psi(u_1, u_2) = \frac{1-u_1^2}{2} + \frac{1-u_2^2}{2} + D(1+u_1)(1+u_2)$$

where $(u_1, u_2) \in [-1,1]^2$, see Figure 5.10.

We note that:

- the other colour we are likely to see is when $(s_1, s_2) = (1,1)$ which corresponds to yellow;

- as $\psi(u_1, u_2) = \psi(u_2, u_1)$ it follows that results obtained using (m_1, m_2) and (m_2, m_1) will be almost identical;

- reducing the value of D would weaken the dependence of U_1 and U_2 and so leads to figures with four colours. In all of our experiments that there are only ever at most three colours.

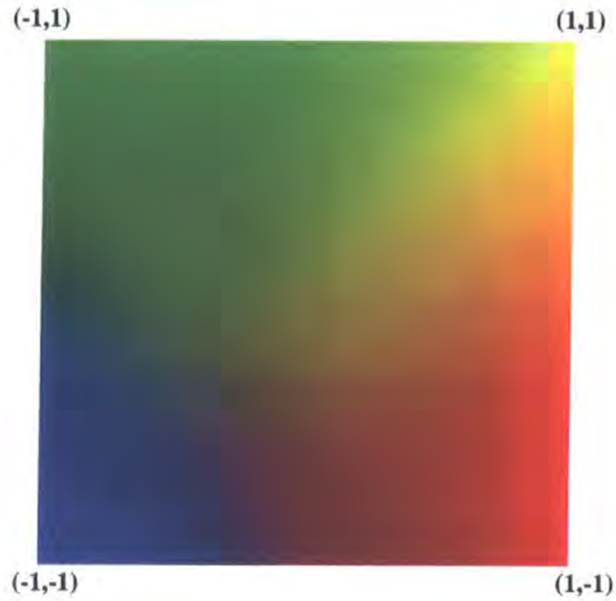


Figure 5.10: Colour key

In the first two simulations we take $(m_1, m_2) = (0, 0)$ and $(-0.5, 0.5)$. In both experiments we find that there are only ever two colours (red and green) and the interface connecting $(1, -1)$ to $(-1, 1)$ does not visit $(-1, -1)$, this remark includes to other simulations not presented here where $m_1 + m_2 = 0$. Moreover, we observe that for $(m_1, m_2) = (0, 0)$, see Figure 5.11, in the early stages of the simulation, a lamellar structure forms and in time these domains grow and shrink as the interfaces migrate and disappear until at the final time we see a stationary composition which consists of three strips and is a two-dimensional analogue of a one-dimensional stationary solution.

In the second experiment the morphology is completely different. Now in the early stages circular domains are observed which grow and shrink and at the final time displays a numerical stationary solution which appears to be a quarter circle.

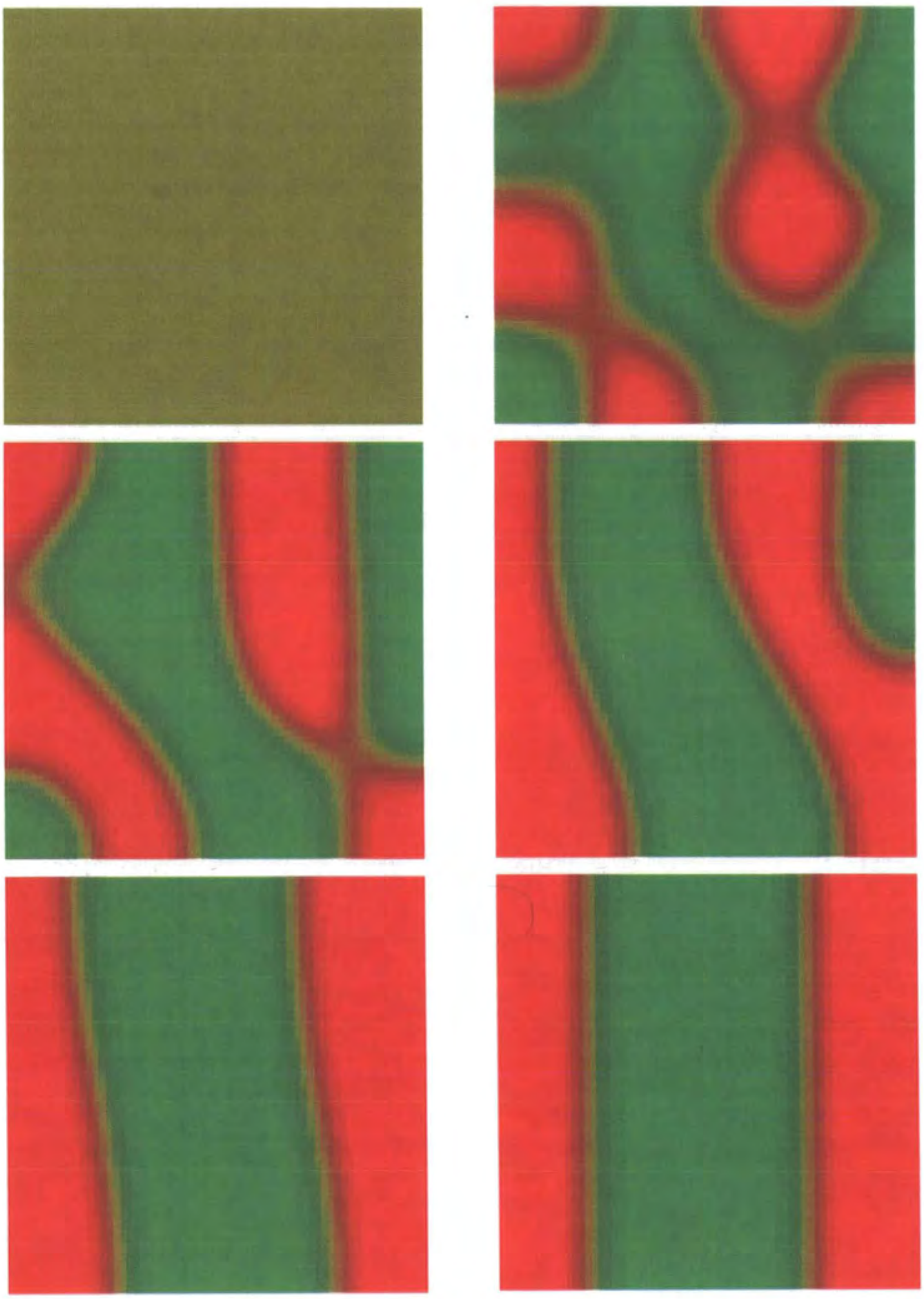


Figure 5.11: Numerical solution where $m_1 = 0$ and $m_2 = 0$ at $t = 0.0156$, 0.1094, 0.1563, 0.2500, 0.5000, 10.0000.

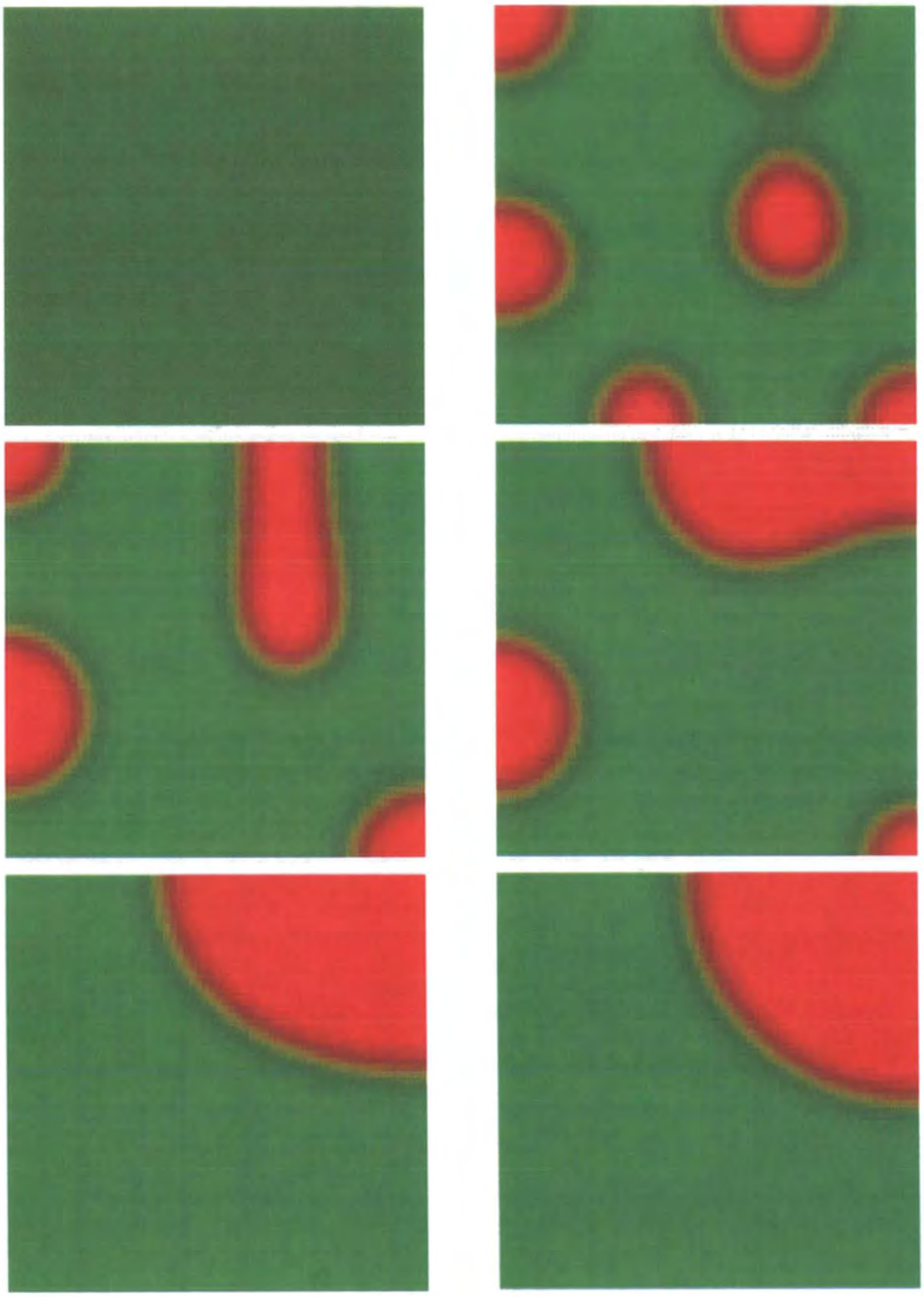


Figure 5.12: Numerical solution where $m_1 = -0.5$ and $m_2 = 0.5$ at $t = 0.0625$, 0.1250, 0.2500, 0.5000, 1.0000, 10.0000.

In the first two experiments we had that $m_1 + m_2 = 0$ and the numerical solution only ever consisted of red and green. In the third experiment we set $m_1 = m_2 = -0.5$ and again we see circular domains in the early stages and at the final time we display a numerical stationary solution. Note the presence of three colours.

In the fourth and fifth experiments we set $m_1 = -0.25$, $m_2 = +0.5$ and $m_1 = -0.5$, $m_2 = +0.25$ so that the numerical solution should be quite similar to the second experiment where $m_1 = -0.5$ and $m_2 = 0.5$. What is of interest is that the phases $(1, 1)$ or $(-1, -1)$ prefers to wet the interface where there is a transition from $(-1, 1)$ to $(1, -1)$ which is also observed at an earlier time in the second experiment and is supported by our tentative conclusion at the end of Section 5.3.1. Although it appears that the region which is blue is thicker in the final experiment Figure 5.15 than the earlier experiment Figure 5.14 on closer investigation this is not the case.

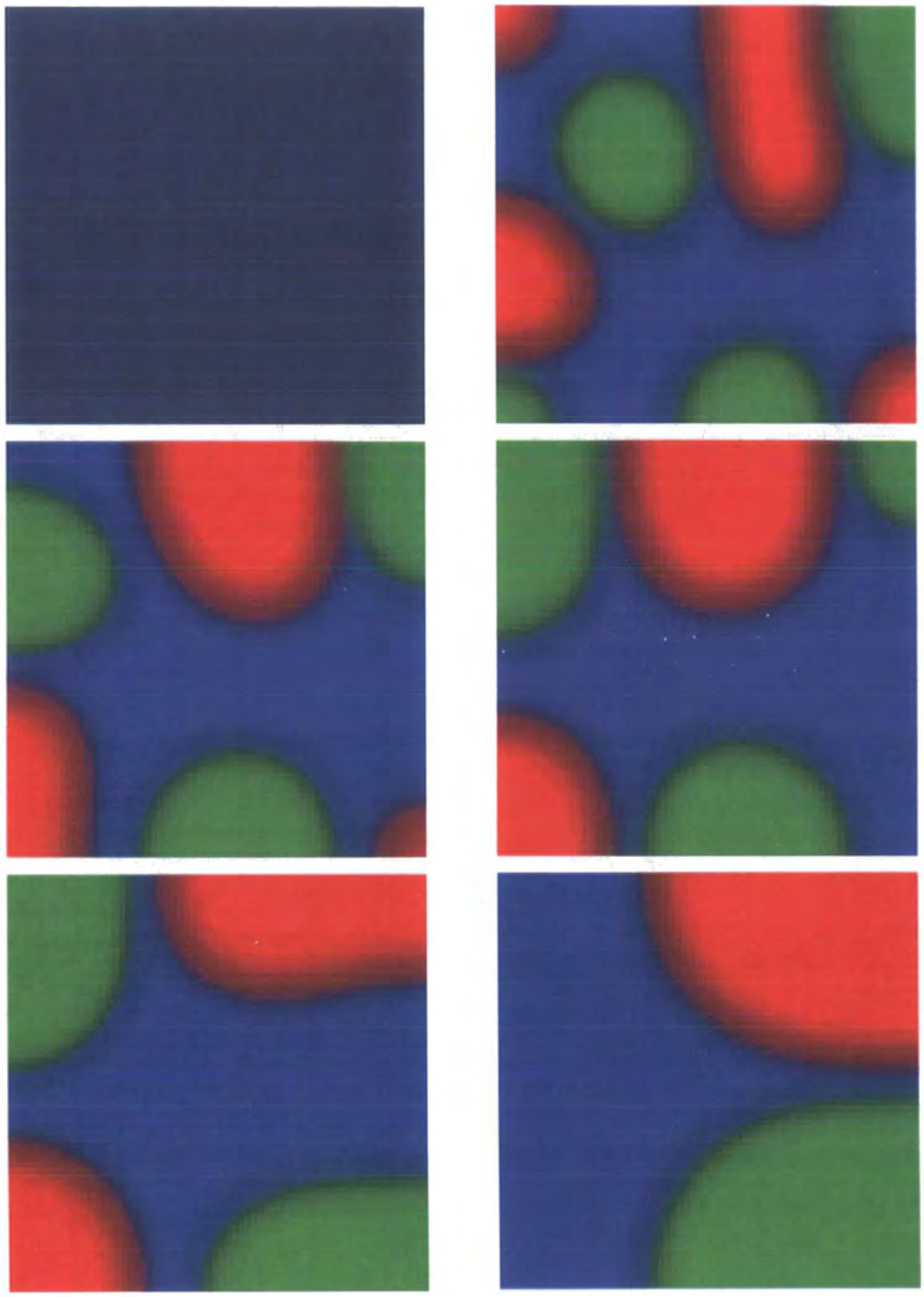


Figure 5.13: Numerical solution where $m_1 = -0.5$ and $m_2 = -0.5$ at $t = 0.0156$, 0.2500, 0.5000, 0.7500, 1.2500, 10.0000.

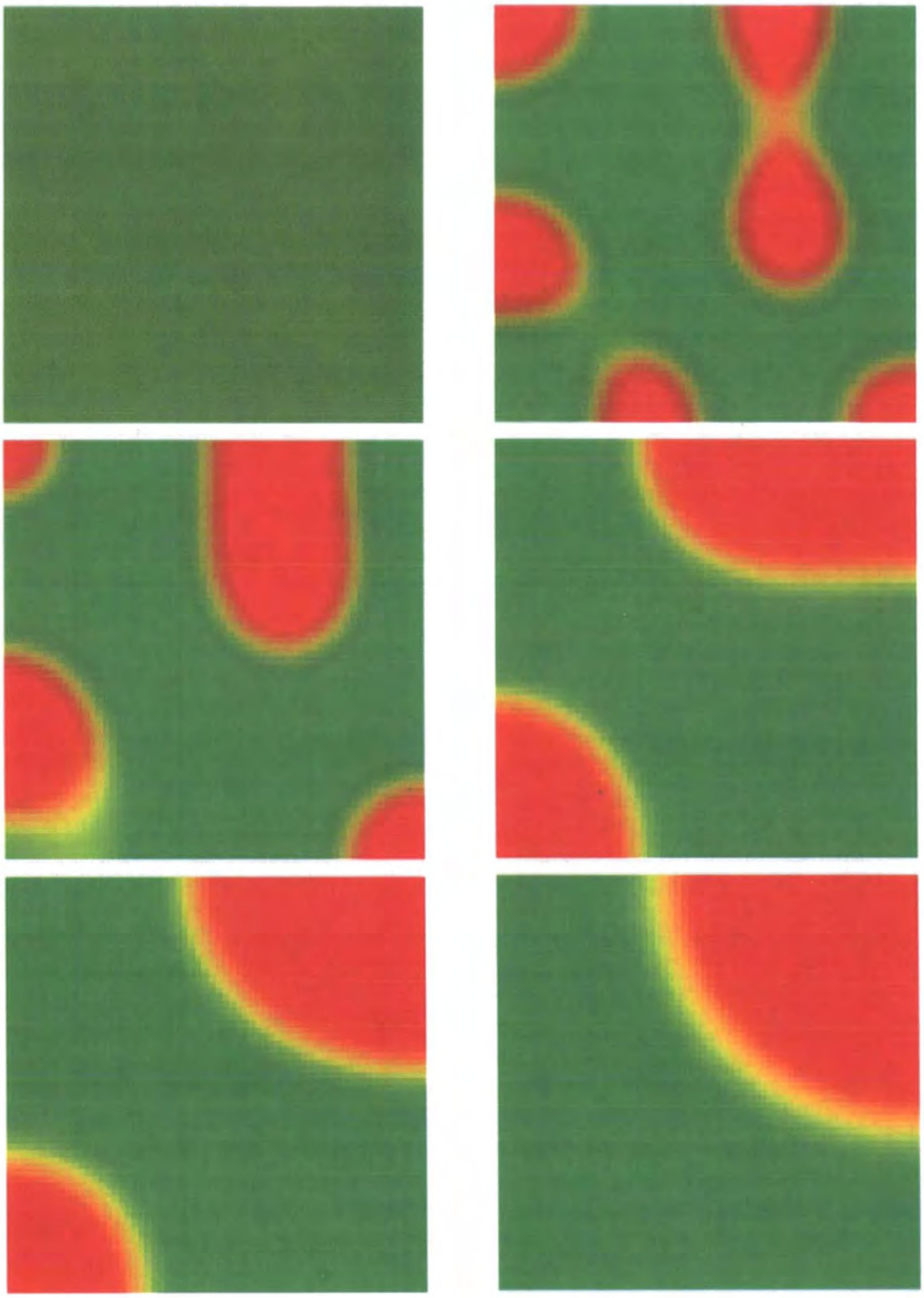


Figure 5.14: Numerical solution where $m_1 = -0.25$ and $m_2 = 0.5$ at $t = 0.0625$, 0.1250, 0.2500, 0.5000, 1.0000, 10.0000.

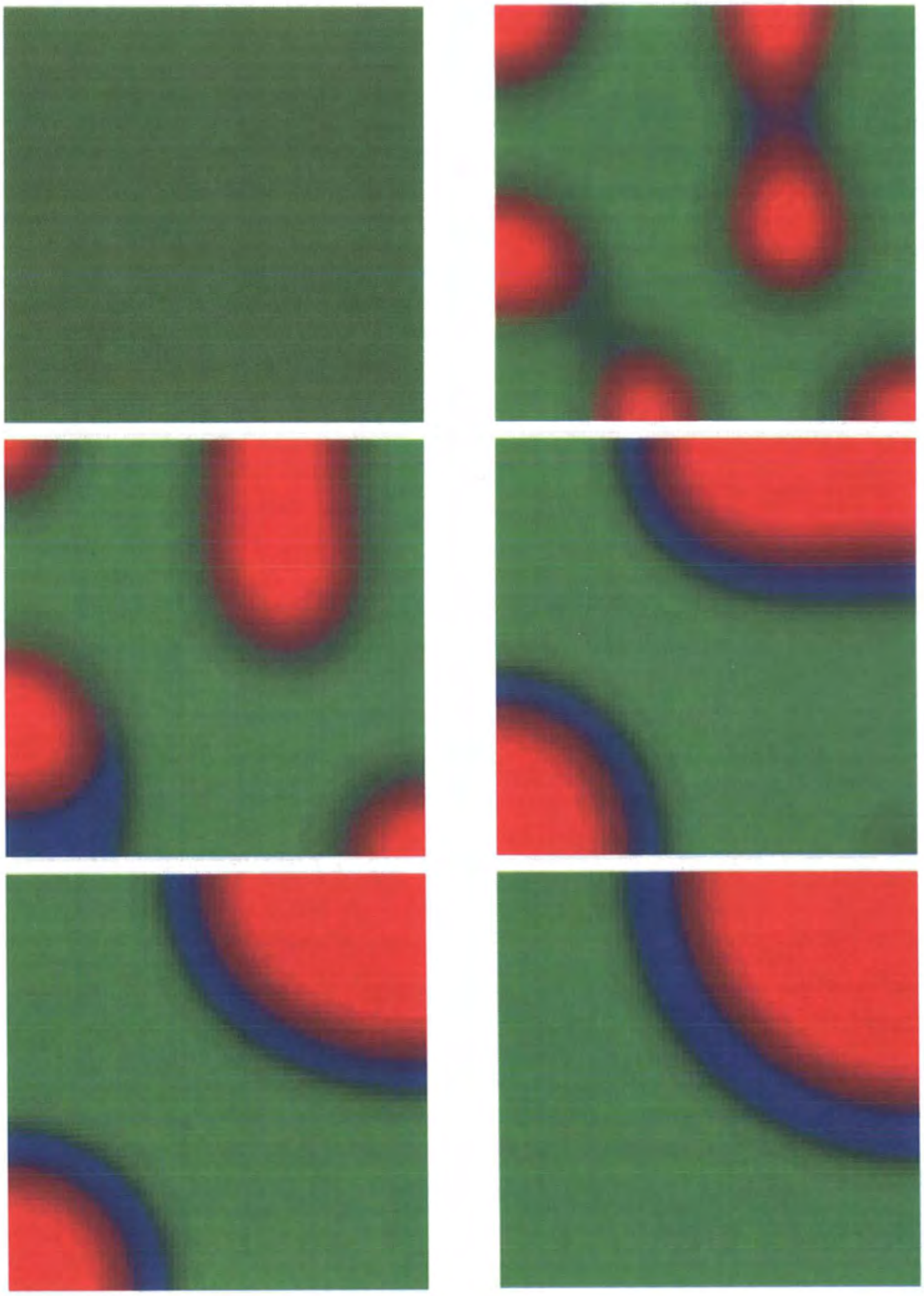


Figure 5.15: Numerical solution where $m_1 = -0.5$ and $m_2 = 0.25$ at $t = 0.0625$, 0.1250, 0.2500, 0.5000, 1.0000, 10.0000.

Chapter 6

Conclusions

The existence and uniqueness of the solution for the coupled pair of continuous Cahn-Hilliard equations modelling a *deep quench* phase separation on a thin film of binary liquid mixture coating substrate, which is wet by one component, has been shown by using a penalty method and a Faedo-Galerkin approximation.

The semidiscrete approximation was proposed. The existence and uniqueness for the semidiscrete finite element have been proven for $d \leq 3$ using a penalised problem. The error bound between the semidiscrete and continuous solution has been presented for $d \leq 3$.

The fully discrete approximation for solving the weak formulation has been expressed. The existence and uniqueness have been proven for $d \leq 3$. The stability and extra stability have been given for $d \leq 3$. The error estimate between the continuous problem and the fully discrete approximation has been given by combining the error bound between the continuous solution and the semidiscrete approximation and the error bound between the semidiscrete and the fully discrete approximations.

The practical algorithm, based on Lions-Mercier spitting algorithm, has been discussed. The existence, uniqueness, and convergence propeties of this algorithm have been proven. The linear stablility analysis for one space dimention was also presented. Simulations in one and two space dimentions have been performed.

Further areas for investigation are:

The analysis of (1.1.16a-d) with the prototype logarithmic free energy

$$
\begin{aligned}
&F(u_1, u_2) \\
&= \frac{T}{2}\Big[(1+u_1)\ln(1+u_1) + (1+u_2)\ln(1+u_2) + (1-u_1)\ln(1-u_1) + (1-u_2)\ln(1-u_2)\Big] \\
&\quad + \frac{T_c}{2}\Big[(1-u_1^2) + (1-u_2^2)\Big] + \frac{\gamma}{2}\Big[|\nabla u_1|^2 + |\nabla u_2|^2\Big] + D(u_1+1)(u_2+1).
\end{aligned}
$$

would be used of great interest.

Moreover, one could attempt to classify the stationary solutions:

Find $\{u_i, \lambda_i\} \in K^h_{m_i}$ such that

$$
\gamma(\nabla u_i, \nabla\eta - \nabla u_i) - (u_i, \eta - u_i) + (D(u_j+1) + \lambda_i, \eta - u) \;\geq\; 0 \;\forall \eta \in K^h.
$$

Also one could look at the γ-limit of such problems as mentioned in the previous chapter.

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Appendix A

Definitions and Auxiliary Results

Definition A.0.1 (almost everywhere, *a.e.*, see Evans [28] p.646)
A property is said to hold 'almost everywhere' (*a.e.*) in Ω (or, for almost every (*a.e.*) x in Ω) if the property is true for all $x \in \Omega \backslash \Gamma$, where Γ is a subset of Ω with measure zero.

Definition A.0.2 (Convex Set and Convex Functional, see Johnson [34] p.249)
Let V be a Hilbert space with scalar product $(\cdot, \cdot)_V$ and norm $\|\cdot\|_V$. A set K is said to be *convex* if for all $u, v \in K$ and $0 \leq \lambda \leq 1$, one has

$$\lambda u + (1 - \lambda)v \in K. \tag{A.0.1}$$

We said that $F : K \mapsto \mathbb{R}$ defined on the convex set K is *convex functional* if for all $u, v \in K$ and $0 \leq \lambda \leq 1$, one has

$$F(\lambda u + (1 - \lambda)v) \leq \lambda F(u) + (1 - \lambda)F(v). \tag{A.0.2}$$

Definition A.0.3 (V-elliptic, see Johnson [34] p.50)
A bilinear $a(\cdot, \cdot)$, on $V \times V$, is said to be V-elliptic if there is a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|^2 \quad \forall v \in V,$$

where V is a Hilbert space with scaler product $(\cdot, \cdot)_V$ and corresponding norm $\|\cdot\|_V$.

Theorem A.0.4 (Convergence properties, see Rodrigues [46] pp.55-56)
Let E be a Banach-space and $\{v_\eta\}$ a sequence:

(i) $v_\eta \rightharpoonup v$ in $E \Leftrightarrow \langle l, v_\eta \rangle \to \langle l, v \rangle, \quad \forall l \in E'$;

(ii) $v_\eta \to v$ (strong) $\Rightarrow v_\eta \rightharpoonup v$ (weak);

(iii) $v_\eta \rightharpoonup v$ in $E \Rightarrow \|v_\eta\|$ is bounded and $\|v\| \leq \liminf \|v_\eta\|$;

(iv) $v_\eta \rightharpoonup v$ in E, $l_\eta \to l$ in E' (strong) $\Rightarrow \langle l_\eta, v_\eta \rangle \to \langle l, v \rangle$

(v) the weak and strong topologies coincide $\Rightarrow E$ has finite dimension

(vi) if $C \subset E$ is convex, C is strongly closed

(vii) if $L: E \to F$ is a linear continuous operator between two Banach spaces it is also continuous from E -weak into F -weak and conversely.

Theorem A.0.5 (Compactness Theorem, see Lions [37] p.58)
Let V, H, V' be three Banach spaces with V and V' being reflexive and

$$V \subset H \equiv H' \subset V',$$

where the injection $V \hookrightarrow H$ is compact. Also let

$$W = \{v : v \in L^{p_0}(0,T;V), \frac{dv}{dt} \in L^{p_1}(0,T;V')\},$$

where $T < \infty$ and $1 < p_0,\ p_1 < \infty$, Then the injection of W in $L^{p_0}(0,T;H)$ is compact.

Proof: See Lions [37] p.58.

Theorem A.0.6 (See Dautray and Lions [24] p.289)
Let V be a reflexive Banach space, $\{\eta_n\}$ a bounded sequence in V. Then it is possible to extract from $\{\eta_n\}$ a subsequence which convergences weakly in V.

Theorem A.0.7 (See Dautray and Lions [24] p.291)
Let V be a separable normed space and V' its dual. Then from every bounded sequence in V', it is possible to extract subsequence which is weak-star convergent in V'.

Theorem A.0.8 (Grönwall Inequality)

Let C be a nonnegative constant and let u and v be continuous nonnegative functions on some interval $t \in [\infty, \beta]$ satisfying the inequality

$$v(t) \leq C + \int_{\infty}^{t} v(s)u(s)\, ds \quad \text{for } t \in [\infty, \beta].$$

Then

$$v(t) \leq C \exp\left(\int_{\infty}^{t} u(s)\, ds\right) \quad \text{for } t \in [\infty, \beta].$$

Proof: See Brauer and Nohel [15] p.31. □

Theorem A.0.9 (Lax-Milgram Lemma)

Let V be a Hilbert space, let $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ be a continuous V-elliptic bilinear form, and let $f : V \mapsto \mathbb{R}$ be a continuous linear form. Then the abstract variational problem: Find an element u such that

$$u \in V \;\text{ and }\; \forall v \in V, a(u,v) = f(v),$$

has one and only one solution.

Proof: See Ciarlet [21] p.8. □

Theorem A.0.10 (Differential Identity)

Let V, H, and V' be three Hilbert spaces, having the property that

$$V \subset H \equiv H' \subset V'.$$

If $u \in L^2(0,T;V)$ and $u' \in L^2(0,T;V')$ then $u \in C([0,T];H) a.e$ and the folling equality holds in the scalar distribution sense on $(0,T)$

$$\frac{d}{dt}|u|^2 = 2\langle u', u\rangle.$$

Proof: See Témann [49] p.261. □

Appendix B

Faedo-Galerkin Method

Faedo-Galerkin Method (Lions [37], Garvie [30])

1. Assume we have a set $\{z_i\}_{i=1}^{\infty}$ of linear independent elements of $H^1(\Omega)$ (or $H_0^1(\Omega)$) such that the linear span of the z_i is dense in $H^1(\Omega)$ (or $H_0^1(\Omega)$). A Galerkin approximation $u^k(\cdot,t) = \sum_{i=1}^{k} c_{ik}(t) z_i(\cdot)$ is substituted into the finite dimensional weak form of the PDE to give a system of k ODEs (an IVP) for $c_{ik}(t)$. Standard ODE theory then gives local existence (and uniqueness) of the $c_{ik}(t)$ and hence of the approximate solution u^k on the finite time interval $(0, t_k)$, $t_k > 0$. This relies on the (local) Lipschitz continuity of the nonlinearities on the right-hand side of the system of ODEs.

2. We deduce that the functions u^k are uniformly bounded with respect to some norm, i.e., $\|u^k\| \leq C$. This bound is called an "*a priori* estimate". Then $t_k = T$ is independent of k, that is we have global existence of u^k.

3. We use "weak compactness" arguments to extract a convergent subsequence (in some sense) from the uniformly bounded sequence of functions. This process is called "passage to the limit". We must also show passage to the limit of each finite dimensional term in the ODE (or, each term in the finite dimensional weak form). It is typically the nonlinear term that gives the most difficulty in the process. This leads to global existence of the weak solution u.

4. To obtain uniqueness of the weak solutions assume there are two weak solutions

u_1 and u_2 with the same initial data. Substract the weak form for u_2 from the weak form for u_1, let the test function $\eta = u_1 - u_2 =: w$ and bound w in terms of the initial data. The aim is to deduce that $w = 0$, i.e., there is one and only one weak solution. If the initial data of the weak solutions u_1 and u_2 are assumed different, then this process leads to continuous dependence of the weak solution on the initial data.